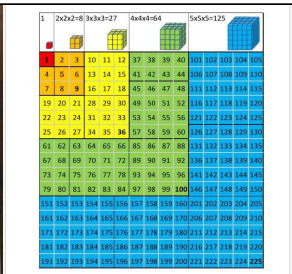


Math Circle 2022-2023 Cyrus, Marc, Dar



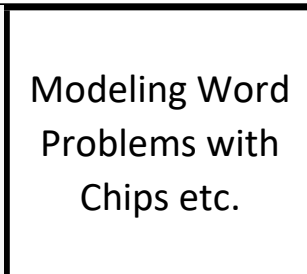
5x5 Grid Problems



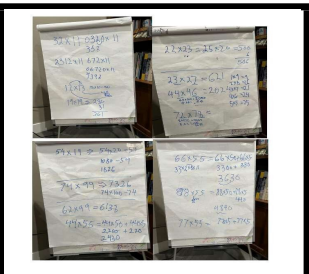
Sums of Squares & Cubes



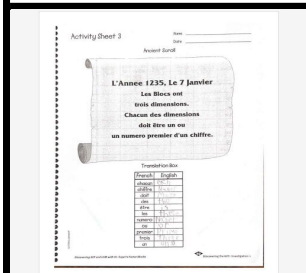
1 <= 2 Machine & Binary Codes



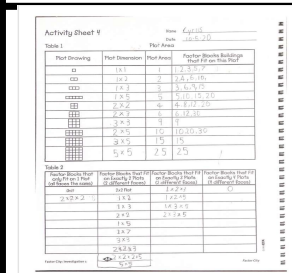
Modeling Word Problems with Chips etc.



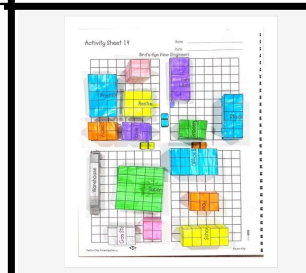
Double Digit Multiplication



Factor Blocks Introduction



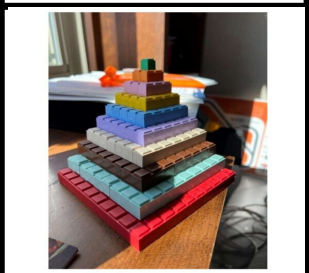
Factor City Powers of Ten



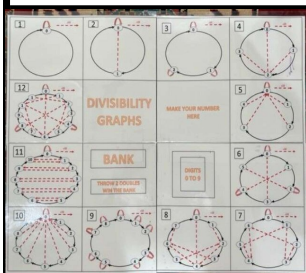
Factor City Completion



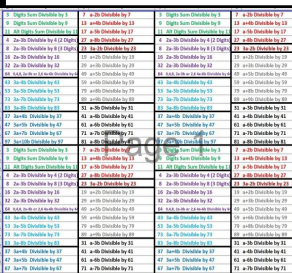
Divisibility Rules Game



Divisibility Rules Explanations



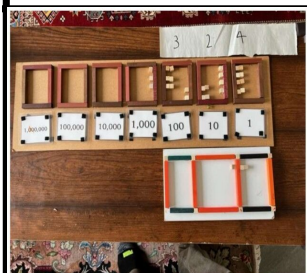
Divisibility Graphs Game



Balls in a Triangle Pyramid



Divisibility Rule to 100 M Game



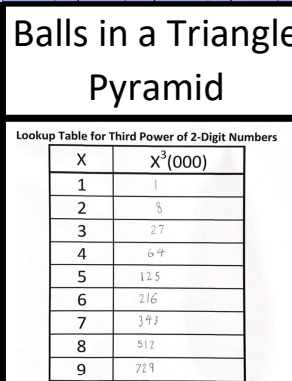
1<==10 Machine



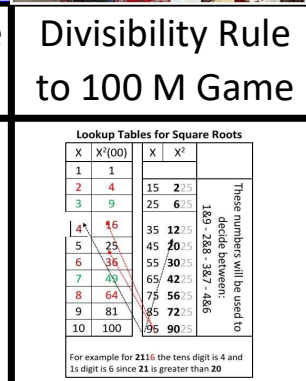
1<==X Machine



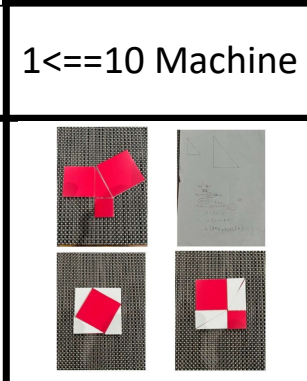
Playing with Edison



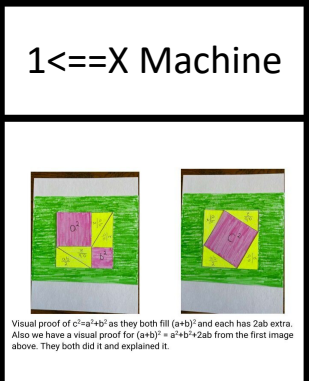
Cube Roots for 2-D Numbers



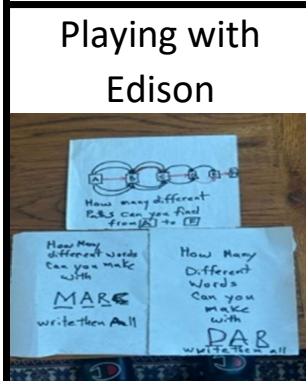
Square Roots for 2-D Numbers



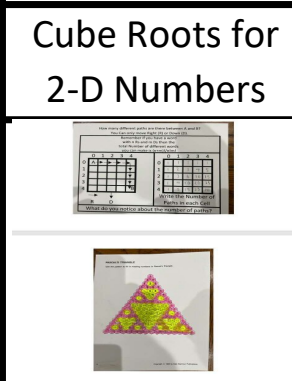
Pytago. Thrm & Triplets - V. Tool



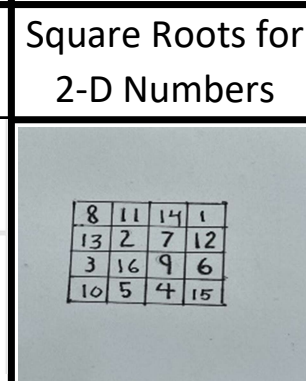
Algebra Iden V. Tool Construct



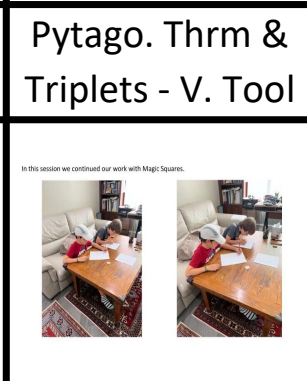
Art of Counting Letters



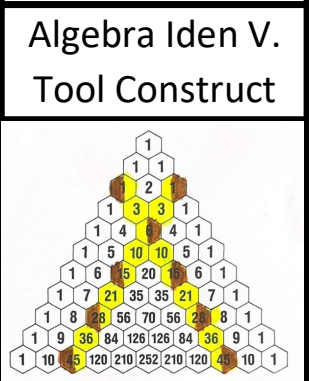
Pascal's Triangle



4x4 Magic Square



Magic Square Manipulation



More Pascal's Triangle

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Math Circle 8/17/2022 2-3 pm

Cyrus, Marc and Dar (Cyrus named me Dar and now everybody calls me Dar)

Dar explained that these sessions will be non competitive and collaborative and we were going to do serious math but enjoy it.

I introduced the 5x5 grid problem as hotel with 25 rooms where all the adjacent (explained it) rooms are connected and at a signal all the guest have to move to an adjoining room. I had made a 5x5 checker board grid and used red and white backgammon pieces to demonstrate the guest with red guest on red squares and white guest on white squares (so that at the end the red guest should all end up in white rooms and the opposite for white guest).

I asked them to try to move the guest and discuss the problem and see if they thought it was possible or not.

I thought I could leave the room and it would take them a few minutes to find a solutions. To my surprise Marc almost immediately said they can not and when I asked him why he correctly pointed to a red guest and said he can not move and made a gesture that showed he had moved the rest in his head. To engage them to work together I said maybe they can work on it a bit more and move the guests and they stated on it. I left the room and came back in a few minutes and they said they were sure it could not be done and showed some of the moves and there was also a red guest that could not move. I pushed further and asked them how many guest in total and tried to get them to say that there are 13 red guest and 12 white rooms so not all the red guest could move to a white room. They went back and forth and finally Cyrus said that there are 13 red guest and then (I should not have) I explained that each red guest has to move to a white room. They were both very engaged;=>

Then I did the variation that if one guest does not move could all the guest move and they both came up after some back and forth that if a red guest stays put this is possible and not if a white guest stays put.

[illegible]

We had a 5 minute break and then restarted

I asked them about the sum of odd numbers (Cyrus said yes this was the Goldbach conjecture) I said that that was really different but explained what a conjecture was and about the sum of consecutive numbers. Then he said oh ya they are always a square. I had them work together so $1+3$ was 4 then plus 5 then 9 then plus 7 was 16 and plus 9 was 25. I asked them to take turns and color on the big board on a 5×5 grid starting with 1 to see if they could see the squares. They did it and liked it (Cyrus had seen this before)

Then I had them find Galileo's ratio of $\frac{1}{3}$ (I had them first find 3 then $\frac{1}{3}$) Cyrus added $1+3$ to get 4 and Marc added $5+7$ to get 12 divided the two numbers $12/4$ to get 3 Cyrus did the opposite and some how got $\frac{1}{8}$ then $\frac{1}{3}$. (They both drew square even though I asked them just to add the numbers!) we continued and they got 9 and 27 and then 16 and 48 and every time 3 and $\frac{1}{3}$. <they were impressed but not as impressed as I was when I saw this yesterday;=> I said they can start on any odd number and Cyrus said 21 but it quickly was evident that $21+23$ would not be $\frac{1}{3}$ of $25+27$ Cyrus said and Mark agreed that we must start with 1 for this to work.

I asked them if they could use the 5x5 grid and show this. Cyrus jumped and said that the square for $1+3$ is really $\frac{1}{4}$ of the square for $1+3+5+7$ so we can see that $5+7$ are made up of 3 2x2 squares. He actually put borders on these 2x2 squares. Then Marc showed the next one on a 6x6 grid with my help) and we concluded that as we went this will continue to happen (Cyrus was very into this and they both seemed to like the visual proof) I explained wat a visual (heuristic) proof was and to do this in math rigorously we need a better proof.

We ended with me telling them about Galileo (I had planted the seed in the car coming over that we would talk about him)

I aske them about dropping a feather and a rock together and which one will hit the ground first. Marc said the stone and when I aske why he said because stone weighed more I said but why. Cyrus jumped up and demonstrated with a piece of cardstock and a pencil and we discussed some more and eventually we got to the fact that air is really what slows down the feather and the paper more than stone and pencil. We also talked about gravity and thru vacuum cleaner got to vacuum and outer-space and spaceships and no gravity (Marc mentioned this that every thing floats there) and the fact that in vacuum all object will fall at the same rate and I explained a little about friction and air friction and spaceships heating up (Cyrus mentioned the atmosphere)

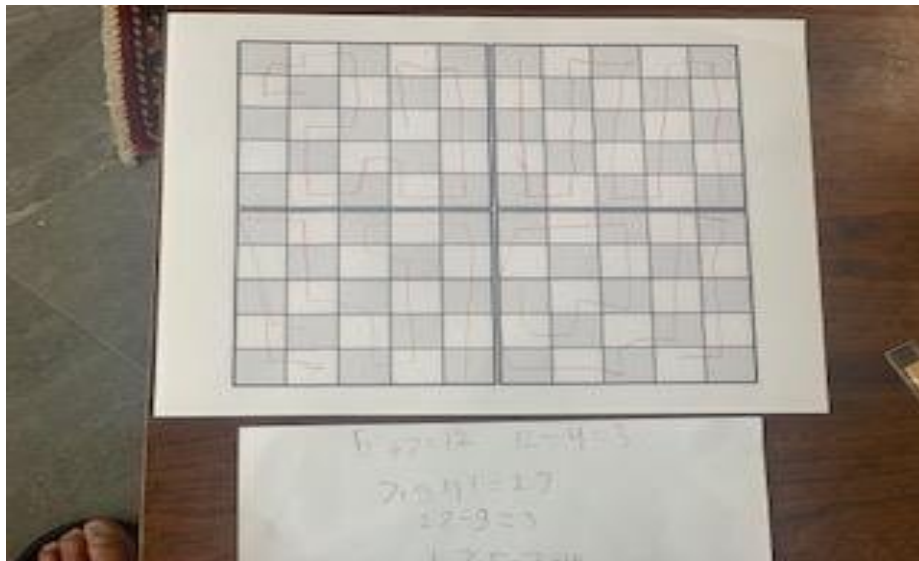
I also told them about how the church harassed Galileo and made him take back what he had said about Earth not being at the center of the universe and we ended there.

I had high hopes for the Math Circle but the session surpassed all my expectations these two boys are amazing and together they are incredible;=> IT was such a joy to see them work together.

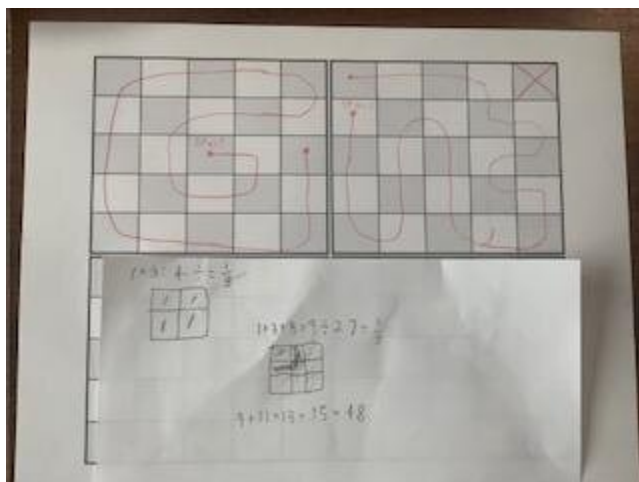
References

For the first sessions I wanted to propose this problem that they can do with a 5x5 board and backgammon pieces. [Thinking 5x5](#) and depending on where they get maybe cover some of the other related problems (Pages 12-17 form the Power of Mathematical Visualization that is attached) There is also the original problem that James Tanton discusses in his first lecture which is vey similar to the problem in the 5x5 Thinking. [The Power of mathematical Visualization.](#)

Marc Work



Cyrus Work



Math Circle 8/24/2022 2-3 pm

Cyrus, Marc and Dar

Reviewed Galilean Fraction using the chart that shows that odd numbers make squares. Cyrus said the Ratio was 3 and when I said it must be a Ratio Marc said and they agreed that it was 1/3. They could see it on the table <but I think I jumped in too much should have shut up and let them explain it;=<

Then I told them about Karl Fredrich Gauss that found the sum of 1 to 100 when he was about their age <and almost as smart;=> I asked them what do they think it was after some back and forth Cyrus came up and Marc agreed that it was 5050 (Cyrus has seen these more than a year ago and it was great that he could remember) I had the laminated formulas on the library sides and he kept saying why they were there. Then I had them take turns and fill the sum for n from 1 to 5 (see figure) and they continued with larger numbers up to a 100. They worked with manipulatives and numbers 1 to 5. And made n by (n+1) rectangles using the manipulatives and going back to the paper work for the 1x2, 2x3, 3x4, 4x5 and 5x6 (see figure). They were very much engaged and I kept my trap shut pretty well and only asked questions and gave them tasks and no real hints. At this point I think they had a good idea that $n(n+1)/2$ was the sum of 1 thru n always. Then we looked at the 5x5 Grid and saw that $1+2+3+4+5+4+3+2+1$ was 25 or 5^2 (and I explained to Mark what power was). Then they placed the manipulatives on the 5x5 grid (I just got them started but could have kept out more) and they really enjoyed it (see figure. Next I showed them that if we added another 5 then they would have the 5x6. Cyrus said this was one of his discoveries and I realized he was right he had found consecutive step squads (this was the name for Number Blocks on you tube) 1 thru 5 and 1 thru 4 make a square! Then I showed them the method that Gauss has supposedly used by adding $100+1,99+2$ and to see there will be 50 of them so $50 \times 101 = 5050$ (see figure).

Then we had a break.

I gave them cubes (these are really historic as I told you from Dienes for 1960s) from 1 to 5 and also flat pieces so they took turns and reconstructed the cubes with the flat pieces (see figure) and I got them started and they took turns to make squares with the flat pieces again taking turns so $1+8$ became a 3×3 square and $1+8+27$ became a 6×6 square up to $1+8+27+64+125$ that became 15×15 square (see figure) they put the Cubes for 1,2,3,4,5 on the two sides of 225 square and saw that this was equivalent to the sum of the cubes from 1 to 125. I showed them that now they had found that $1^3 + 2^3 + 3^3 + 4^3 + 5^3 = (1+2+3+4+5)^2$ so generally

$$1^3 + 2^3 + 3^3 + \dots + n^3 = [(n(n+1)/2)]^2$$

Cyrus at this point grabbed the laminate sheet I had made for Mark and gave it to him. He was disappointed that he could not find the one that went 10 but we shall locate it and include it here. We looked at the sheet and they liked the fact that they could see the relation there. I am pretty sure they really got the fact that this formula will always work.

Cyrus then wanted to show Mark his Reversible Square numbers 12 and 13 that when they are reversed and squared has squares that the reverse of original numbers 144 and 441 and 169 and 961 Marc was impressed (I was originally very impressed as he had found this out of the blue before he was 7) Cyrus also did his trick with base 2 number and in Scratch with the Magical Mind Reader. Mark said that he

had seen this (Cyrus had taken a cube to school last year) but he showed interest (Marc is cool that way).

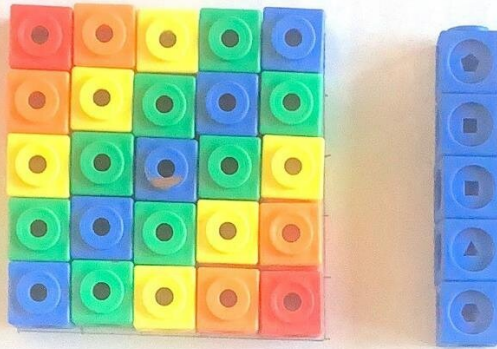
We stopped here their chemistry of working together is great I just have to be more quiet and give them even more time to hash out things together. I was going to give them some science stuff but decided that they had enough for one session.

$$\sum_{m=1}^n m = \frac{n \times (n+1)}{2} = 1 + 2 + 3 + \dots + n$$

n	n+1	n x (n+1)	$\frac{n \times (n+1)}{2}$
1	2	1 x 2 = 2	$\frac{2}{2} = 1$
2	3	2 x 3 = 6	$\frac{6}{2} = 3$
3	4	3 x 4 = 12	$\frac{12}{2} = 6$
4	5	4 x 5 = 20	$\frac{20}{2} = 10$
5	6	5 x 6 = 30	$\frac{30}{2} = 15$

n	n+1	n x (n+1)	$\frac{n \times (n+1)}{2}$
10	11	10 x 11 = 110	$\frac{110}{2} = 55$
20	21	20 x 21 = 420	$\frac{420}{2} = 210$
50	51	50 x 51 = 2550	$\frac{2550}{2} = 1275$
100	101	100 x 101 = 10100	$\frac{10100}{2} = 5,050$

$$1+2+3+4+5+4+3+2+1 = 25 = 5^2 = 5 \times 5$$



$$\underbrace{1+2+3+4+5}_{15} + \underbrace{1+2+3+4+5}_{15} = 30 = 25+5$$

$$\frac{100+10}{2} = \frac{110}{2} = 55$$

$$\frac{(n)(n+1)}{2} = \frac{n^2+n}{2}$$

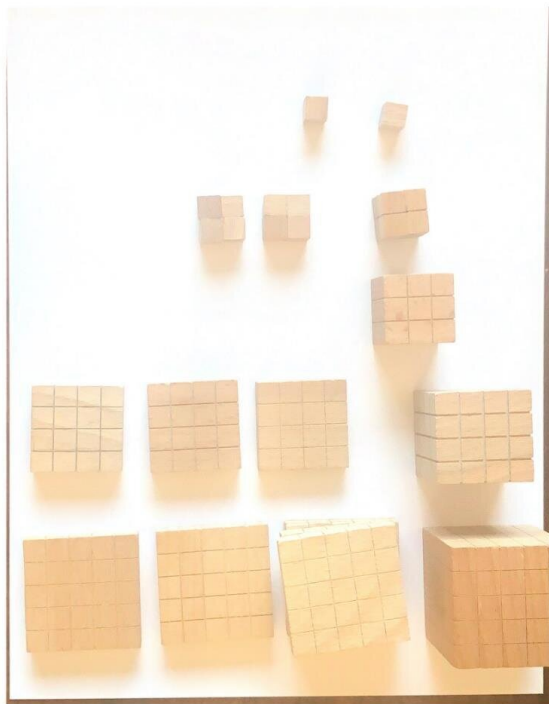
$$\begin{array}{r} 1 \\ 2 \\ 3 \\ \hline \end{array} \cdot \begin{array}{r} 50 \\ 51 \\ \hline 10 \end{array}$$

$$50 \times 101 = 5050$$

$$\begin{array}{r} 99 \\ 100 \\ \hline 199 \end{array}$$

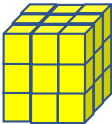
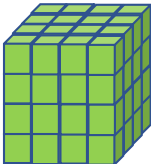
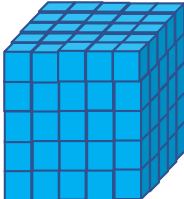
$$101 = 51 \times 2$$

$$100 = 50 \times 2$$



$$1^3 + 2^3 + 3^3 + 4^3 + 5^3 = (1+2+3+4+5)^2$$

$$1^3 + 2^3 + \dots + n^3 = (1+2+\dots+n)^2 = \left(\frac{n(n+1)}{2}\right)^2$$

1	2x2x2=8		3x3x3=27			4x4x4=64				5x5x5=125				
														
1	2	3	10	11	12	37	38	39	40	101	102	103	104	105
4	5	6	13	14	15	41	42	43	44	106	107	108	109	110
7	8	9	16	17	18	45	46	47	48	111	112	113	114	115
19	20	21	28	29	30	49	50	51	52	116	117	118	119	120
22	23	24	31	32	33	53	54	55	56	121	122	123	124	125
25	26	27	34	35	36	57	58	59	60	126	127	128	129	130
61	62	63	64	65	66	85	86	87	88	131	132	133	134	135
67	68	69	70	71	72	89	90	91	92	136	137	138	139	140
73	74	75	76	77	78	93	94	95	96	141	142	143	144	145
79	80	81	82	83	84	97	98	99	100	146	147	148	149	150
151	152	153	154	155	156	157	158	159	160	201	202	203	204	205
161	162	163	164	165	166	167	168	169	170	206	207	208	209	210
171	172	173	174	175	176	177	178	179	180	211	212	213	214	215
181	182	183	184	185	186	187	188	189	190	216	217	218	219	220
191	192	193	194	195	196	197	198	199	200	221	222	223	224	225

1	2x2x2=8	3x3x3=27	4x4x4=64	5x5x5=125	6x6x6=196	7x7x7=343	8x8x8=512	9x9x9=729	10x10x10=1,000																																													
																																																						
1	1+8=9	9+27=36	36+64=100	100+125=225	225+196=441	441+343=784	784+512=1296	1296+729=2025	2025+1000=3025																																													
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55
2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	
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Math Circle 3 9/8/2022 3:15-4:15 pm

Cyrus, Marc and Dar

Skipped the session for 9/1/2022 as Cyrus had his operation.

We discussed power and in particular powers of 2 and looked at cards showing powers of 2 from 0 to 12 (see figure). Then Cyrus and Mark examined the $2 \rightarrow 1$ machine (see figure) that can code numbers between 1 to 127 in decimal. I explained that this was called a binary number as it only had zeros and ones. They did the ones and zeros for a few numbers and then proceeded with the challenge of deciphering the code that was given to them by finding the decimal value for each row representing a number between 1 to 127 (see figure). They found all the decimal numbers by making the number on the $2 \rightarrow 1$ machine and then adding the relevant powers of 2 indicated by a 1 in the corresponding square. They worked really well together and would double check the decimal number that they found.

Then I gave them the ASCII table and using it they collaborated to decipher the message; "Cyrus&Marc are Great" They enjoyed the process very much. I explained that this was how computers kept the information and before they literally used light bulbs as 1 when it was on and 0 when it was off and now they used transistors that are very small. We discussed this for a little and I told them that they had another message on the back of the card that they could do later at home if they wanted to. Marc did this at home last night and Cyrus did it this morning and they seem to have been excited about doing it so it was not a burden but a pleasure!

Next we discussed the Magic cards and their secret and how it related to what they had done. They seemed impressed, Marc in particular as Cyrus has seen parts of this before but I think they both really got the connection in particular after we compared the binary representation of a few numbers and the cards on which this number is found. At the end I gave them some challenges (see figure) to find the binary representation of larger numbers using the cards on the floor and they really liked it and found them all. They were a little insulted that I wanted to check their work with a calculator but I did it and they got a kick out of saying we told you it was correct;=>

It was a little after 3 so I decided that we should have fun and we played a few hands of Krypto. Cyrus is really good at Krypto now and beats everybody including me most of the time and I am sure Marc will do the same after a short while.

We stopped here their chemistry of working together is getting even better. I have improved also and give a minimum of instructions and try not to interfere at all. They put in a lot of time deciphering the binary message but seemed to enjoy doing it which was my goal for the activity.



Find the Decimal Code and then
Decipher the Message by Finding the ASCII Characters

ASCII Character	Decimal	128	64	32	16	8	4	2	1
T	84	0	1	0	1	0	1	0	0
h	104	0	1	1	0	1	0	0	0
e	101	0	1	1	0	0	1	0	1
y	121	0	1	1	1	1	0	0	1
e	32	0	0	1	0	0	0	0	0
a	97	0	1	1	0	0	0	0	1
f	114	0	1	1	1	0	0	1	0
e	101	0	1	1	0	0	1	0	1
e	32	0	0	1	0	0	0	0	0
q	113	0	1	1	1	0	0	0	1
u	117	0	1	1	1	0	1	0	1
i	105	0	1	1	0	1	0	0	1
c	99	0	1	1	0	0	0	1	1
k	107	0	1	1	0	1	0	1	1
L	76	0	0	1	0	0	0	0	0
e	101	0	1	1	0	0	1	0	1
a	97	0	1	1	0	0	0	0	1
r	114	0	1	1	1	0	0	1	0
h	110	0	1	1	0	1	1	1	0
e	101	0	1	1	0	0	1	0	1
r	114	0	1	1	1	0	0	1	0
s	115	0	1	1	1	0	0	1	1

ASCII (As Key) Abbreviated form
American Standard Code for Information Interchange

ASCII Code	Binary Code	ASCII Character Name	ASCII Code	Binary Code	ASCII Character Name
64	01000000	@ At symbol	96	01100000	Grave accent
65	01000001	A Uppercase A	97	01100001	a Lowercase a
66	01000010	B Uppercase B	98	01100010	b Lowercase b
67	01000011	C Uppercase C	99	01100011	c Lowercase c
68	01000100	D Uppercase D	100	01100100	d Lowercase d
69	01000101	E Uppercase E	101	01100101	e Lowercase e
70	01000110	F Uppercase F	102	01100110	f Lowercase f
71	01000111	G Uppercase G	103	01100111	g Lowercase g
72	01001000	H Uppercase H	104	01101000	h Lowercase h
73	01001001	I Uppercase I	105	01101001	i Lowercase i
74	01001010	J Uppercase J	106	01101010	j Lowercase j
75	01001011	K Uppercase K	107	01101011	k Lowercase k
76	01001100	L Uppercase L	108	01101100	l Lowercase l
77	01001101	M Uppercase M	109	01101101	m Lowercase m
78	01001110	N Uppercase N	110	01101110	n Lowercase n
79	01001111	O Uppercase O	111	01101111	o Lowercase o
80	01010000	P Uppercase P	112	01110000	p Lowercase p
81	01010001	Q Uppercase Q	113	01110001	q Lowercase q
82	01010010	R Uppercase R	114	01110010	r Lowercase r
83	01010011	S Uppercase S	115	01110011	s Lowercase s
84	01010100	T Uppercase T	116	01110100	t Lowercase t
85	01010101	U Uppercase U	117	01110101	u Lowercase u
86	01010110	V Uppercase V	118	01110110	v Lowercase v
87	01010111	W Uppercase W	119	01110111	w Lowercase w
88	01011000	X Uppercase X	120	01111000	x Lowercase x
89	01011001	Y Uppercase Y	121	01111001	y Lowercase y
90	01011010	Z Uppercase Z	122	01111010	z Lowercase z
91	01011011	[Opening bracket	123	01111011	{ Opening brace
92	01011100	\ Backslash	124	01111100	Vertical bar
93	01011101	] Closing bracket	125	01111101	} Closing brace
94	01011110	^ Caret - circumflex	126	01111110	~ Tilde
95	01011111	_ Underscore	127	01111111	Delete

Find the Decimal Code and then
Decipher the Message by Finding the ASCII Characters

ASCII Character	Decimal								
		0	1	0	0	0	0	1	1
		0	1	1	1	1	0	0	1
		0	1	1	1	0	0	1	0
		0	1	1	1	0	1	0	1
		0	1	1	1	0	0	1	1
		0	0	1	0	0	1	1	0
		0	1	0	0	1	1	0	1
		0	1	1	0	0	0	0	1
		0	1	1	1	0	0	1	0
		0	1	1	0	0	0	1	1
		0	0	1	0	0	0	0	0
		0	1	1	0	0	0	0	1
		0	1	1	1	0	0	1	0
		0	1	1	0	0	1	0	1
		0	0	1	0	0	0	0	0
		0	1	0	0	0	1	1	1
		0	1	1	1	0	0	1	0
		0	1	1	0	0	1	0	1
		0	1	1	0	0	0	0	1
		0	1	1	1	0	1	0	0

Math Circle 3 9/14/2022 1:45 to 3:300 pm

Cyrus, Marc and Dar

We worked on word problems in this session. I think for both Cyrus and Marc it is important to work on coming up with models to solve problems and open ended problems. They really like to solve problems fast, that is OK, but being able to think and model and discuss will help them become good mathematicians and scientists in general.

I used the following problems from the [Power of Mathematical Visualizations](#) with some minor modifications.

I started with this problem

1. In a second grade class $\frac{3}{5}$ of the students are girls and $\frac{2}{5}$ are boys. If $\frac{1}{6}$ of the girls play soccer and $\frac{3}{4}$ of the boys play soccer. What fraction of the class plays soccer?

They both wanted to give an answer quickly. I used chips and encouraged and showed them (as this was the first problem) How this may be done with first starting with 2 chips of one color and 3 of another and then adding another set to be able to see that there will be 4 out of 10 students who will play soccer. I did let them to get in and set up. Cyrus said this was 40% and we had a discussion about percent vs fraction as Mark said he has seen percentages but not worked with them too much.

Then I gave them this similar problem on a piece of paper and let them set it up. They still wanted to give answers before doing the modeling but I encouraged them to model first. And they did it. They found the answer which was again $\frac{4}{10}$ and 40%.

2. In a second grade class $\frac{3}{5}$ of the students are girls and $\frac{2}{5}$ are boys. $\frac{1}{2}$ of the girls are blonde and $\frac{1}{4}$ of the boys are blonde. What fraction of the class is blonde?

We continued with another problem with the same model and they did it quickly and found the answer to be 5 out of 10 and we had to talk some for them to come up that this was $\frac{1}{2}$ or 50%

3. In the same class $\frac{1}{3}$ of the girls speak more than one language but $\frac{3}{4}$ of the boys speak more than one language. What fraction of the class speaks more than one language.

Then I brought one small beaker with measurements and poured in some salt in a pretend 1 liter glass. They wanted to get in and Cyrus poured in a couple of spoon of salt and with mark then we poured in 8 table spoons (this was not exact of course) into a glass with twice as much water. Then we mixed the too and they had to answer the quest. They really liked this and also I helped them model it with chips and they came up with 10 chips out of 30 which was $\frac{1}{3}$ or 33% (give or take) they really enjoyed this problem.

4. If we mix 1 liter of 20% solution of saline (the ratio of the salt molecules to total molecules is 2 to 10) and 2 liters of a 40% saline solution. What will be the percent of salt in the new saline solution (ratio of salt molecules to the total molecules)?

This next problem they got in a flash eight groups 32 students each group has 4 students.

5. We are told that in a class of 32 two students, there are 3 groups of boys and 5 groups of girls. But we are not told what the group size is. Can you find the group size.

I help set up this problem as equations with chips (they liked that) for them but they had trouble deducing that they could take 1 apple and 2 oranges from the first equation and get that one apple and one orange cost \$1.3 it was a little easier for them now to get that one orange must cost \$0.75 and one apple will be \$.55.

6. Together, 2 apples and 3 oranges cost \$3.35. Together, 1 apple and 2 oranges cost \$2.05. What is the price of 1 apple?

Then we took a brake. This first part was about 45 minutes

We took a break for 10 minutes and after the break they did not want to continue with the problems (Cyrus gently told me) I asked them if they wanted to play with Babylonian Fraction Game, and they said yes. So I brought out the cards and we played a few hands with real cards and Cyrus said we should play on the computer (he knows I like him to do that) so we played on the compute. Cyrus actually went there. Marc said he had played before but if he was not quite sure what was going on. Now after the playing with the real cards he really liked it on the computer. Cyrus was going wild at the end (low sugar) but Marc continued. (I really think this is an excellent game for them to see the basics of fraction addition and subtraction and equivalent fractions.

We did not do the following problems that I had also prepared for them.

In this session I think I should have stayed out of the modeling more and given them more time but all in all the session went good.

7. A furniture store buys couches from a wholesaler and marks up the price of each couch 50% for sale in the store. One weekend the store has a sale: All couches $\frac{1}{3}$ off. How uch profit, as a percentage, does the store make on each couch during the sale?
8. Box A contains $\frac{1}{3}$ more coconuts than box B. If half the coconuts are taken from box A and moved to box B, what will be the ratio of the number of coconuts in box A to the number in box B?
9. If it takes 6 women 8 days to build 3 houses, how long will it take 8 women to build 9 houses?

Math Circle 5 9/21/2022 2:00 to 2:45 pm

Cyrus, Marc and Dar

We worked on double digit multiplication and rules for finding cube roots of 2-digit numbers. It was a little shorter session and both Marc and Cyrus had some introduction to these methods and came up with some of their own methods also.

I started with this problem

$32 \times 11 \Rightarrow 0320 \times 11 \Rightarrow 352$ Place 2 zeros at the beginning and the end of the number then add to each of the original digits the digit to its right (right neighbor)

They seem to know both the rule with or without the zeros. I had them write down the method and put it on the board and discuss the method that they had applied.

Then I gave them a 4 digit number multiplied by 11 and they did it no problem and also put it on the board and showed that it is the same algorithm.

$2312 \times 11 \Rightarrow 023120 \times 11 \Rightarrow 25,432$

Without mentioning it I gave them a problem that the sum a number and it's right hand neighbor is greater than 9:

They did know that they had to carry a 1 (10) to the left $672 \times 11 = 06720 \times 11 = 6^1 392 = 7392$

I had the multiplication by 12 that is similar also planned but left it for later.

Then we discussed multiplying numbers between 11 to 19 and again they both could do these fast and they even remembered the results with out using the technique. I had a visual that I gave Marc with more problems to do.

$12 \times 13 = 12 + 3 \times 10 + 2 \times 3 = 150 + 6 = 156$

$16 \times 17 = 272$ $19 \times 19 = 361$ $19 \times 19 = 361$ (Cyrus knew this by heart)

Then I moved to 54×19 and they had to work on it some. I mentioned that one good technique was to do the following and they agreed.

$54 \times 19 = 54 \times 20 - 54 = 1080 - 54 = 1046$

Next I gave them 74×99 and mentioned that a similar technique may be used. They did a little work but came up with

$74 \times 99 \Rightarrow 74 \times 100 - 74 = 7400 - 74 = 7326$

I gave them 62×99 and they had the answer quick but Cyrus first subtracted 99 instead of 66 but reversed himself.

$62 \times 99 = 66 \times 100 - 62 = 6538$

Then we looked at the series of problems with 55 and repeating even numbers 44×55 , 66×55 and 88×55

And they had no trouble coming up with the answers with different methods we discussed the method that I suggested and liked it.

$$44 \times 55 = 44 \times 50 + 44 \times 5 = 2200 + 220 = 2420$$

$$66 \times 55 = 66 \times 50 + 66 \times 5 = 3300 + 330 = 3630$$

$$88 \times 55 = 88 \times 50 + 88 \times 5 = 4400 + 440 = 4840$$

Then I asked them if they recognized any part of the solution and they came up with the fact that it was multiplying something by 11. They even mentioned that there are two 11 factors there and showed them and we discussed that we can factor these numbers both to get an 11 and a 10 factor. They knew that the order of factors did not matter but we did not discuss it (we should have).

$$66 \times 55 = 33 \times 2 \times 5 \times 11 = 33 \times 11 \times 10 = 3630$$

$$88 \times 55 = 44 \times 11 \times 10 = 4840$$

The process was quite interactive and Cyrus wanted 77×55 which I resisted but should not have not, it would have been instructive to see the carrying.

$$77 \times 55 = 77 \times 50 + 77 \times 5 = 3850 + 385 = 4235 = 385 \times 11 = 3^{11} 35 = 4235 \text{ <shame on me;=>}$$

They were ready for a break in particular Cyrus.

After the break I decided to not continue with the multiplication and had Cyrus show the method for doing cube roots of double digit numbers mentally to Marc. He answered a couple that Marc gave him the first really fast the second not as fast as he got confused on the thousands part of the product but got it. He differed to me the explanation after trying some. And I covered them with Marc that will become proficient soon using his cheat sheet. Then I asked him a few with the cheat sheet and we ended.

All in all not a bad session but a little shorter than usual. I may continue with some more rules next week or work with Dr. Super's Factor B locks so they can do GCF and LCM and come back to double digit in another session.

$$32 \times 11 = 352$$

$$2312 \times 11 = 25432$$

$$12 \times 13 = 156$$

$$19 \times 19 = 361$$

$$22 \times 23 = 506$$

$$23 \times 27 = 621$$

$$44 \times 46 = 2024$$

$$72 \times 73 = 5256$$

$$54 \times 19 = 1026$$

$$74 \times 99 = 7326$$

$$62 \times 99 = 6138$$

$$44 \times 55 = 2420$$

$$66 \times 55 = 3630$$

$$88 \times 55 = 4840$$

$$77 \times 55 = 4235$$

$$32 \times 11 = 352$$

023120

25432

06720

7392



$$32 \times 11 = 352$$

because - you just
add 3+2 which = 5
and put the 5 in the
middle.

$$32 \times 11$$

$$0320 \times 11 = 352$$

$$2312 \times 11 = 25432$$

023120
25432

06720
7392

195112

24,399

Math Circle 6 9/28/2022 1:50 to 3:10 pm

Cyrus, Marc and Dar

We worked on Factor Blocks. The activities are from 3 teachers guides that I have co-authored for the use of Factor Blocks. This manipulative was the most successful of my 3 major manipulatives and it had a 10+ year run in schools and it is a shame that was discontinued by ETA. It helps students in grades 4-6 visualize factors and multiplication and eventually Greatest Common Factor GCF and Least Common Multiple LCM. We will spend the next 2 weeks also on these activities and may come back to it for GCF and LCM later on after more work on Fractions.

I told them about Factor Blocks and the fact that I had invented the Manipulatives and showed them the Patent and discussed what it meant. They seemed interested and also I gave them their sets to open and explore.

Cyrus with my help read and translated the scroll that describes the definition of the Factor Blocks in French after they had both manipulated and ordered and reordered the Factor Block finding how many different blocks were in it and how they were called. We discussed the story and also what it meant by looking at the different Factor Blocks.

Next they organized together the Factor Blocks and grouped them in different ways and eventually using one of each kind ordered them from the smallest to largest and explored what their dimensions were and how they should be named and how the Scroll had defined them. The Scroll defined them as "3 Dimensional blocks that have 1 or a one digit prime number for each dimension"

Next they completed Activity Sheet 1 and they discussed the naming of the dimensions and went back and forth to get the correct names. They did this much faster than I thought would take so I also discussed the fact that not all Factor Blocks were in the set and gave them the challenge that I had initially planned to send home with them.

They took more time with this but they seemed to enjoy it and both completed it checking from time to time with each other to make sure they had the right name and the number of Units. They also came up with the fact that since there were 20 more than the original 15 Factor Blocks in this set if they were all included the set would have become way too large for students to use.

Next I told them about the Factor City that they were going to built next time and the fact that the building have footprints and they used the Bird's Eye View Square Paper and the Factor Blocks to draw the foot prints of all the possible Factor Blocks and found that there were 11 different footprints. They colored these foot prints and played with placing the Factor Blocks on them.

I was going to let them use the engineering paper to draw 3-D pictures of some of the Blocks (which is a challenging task initially) but since we had spent 1:20 minutes already I decided to leave that for next time.

Hedi was feeding the boys and Marc had Raspberries that he likes and Cyrus his Almond Butter Jelly. It was quite impressive that they worked for that long and they did not even want a break. A very good session.

ETA[®]

Factor city!

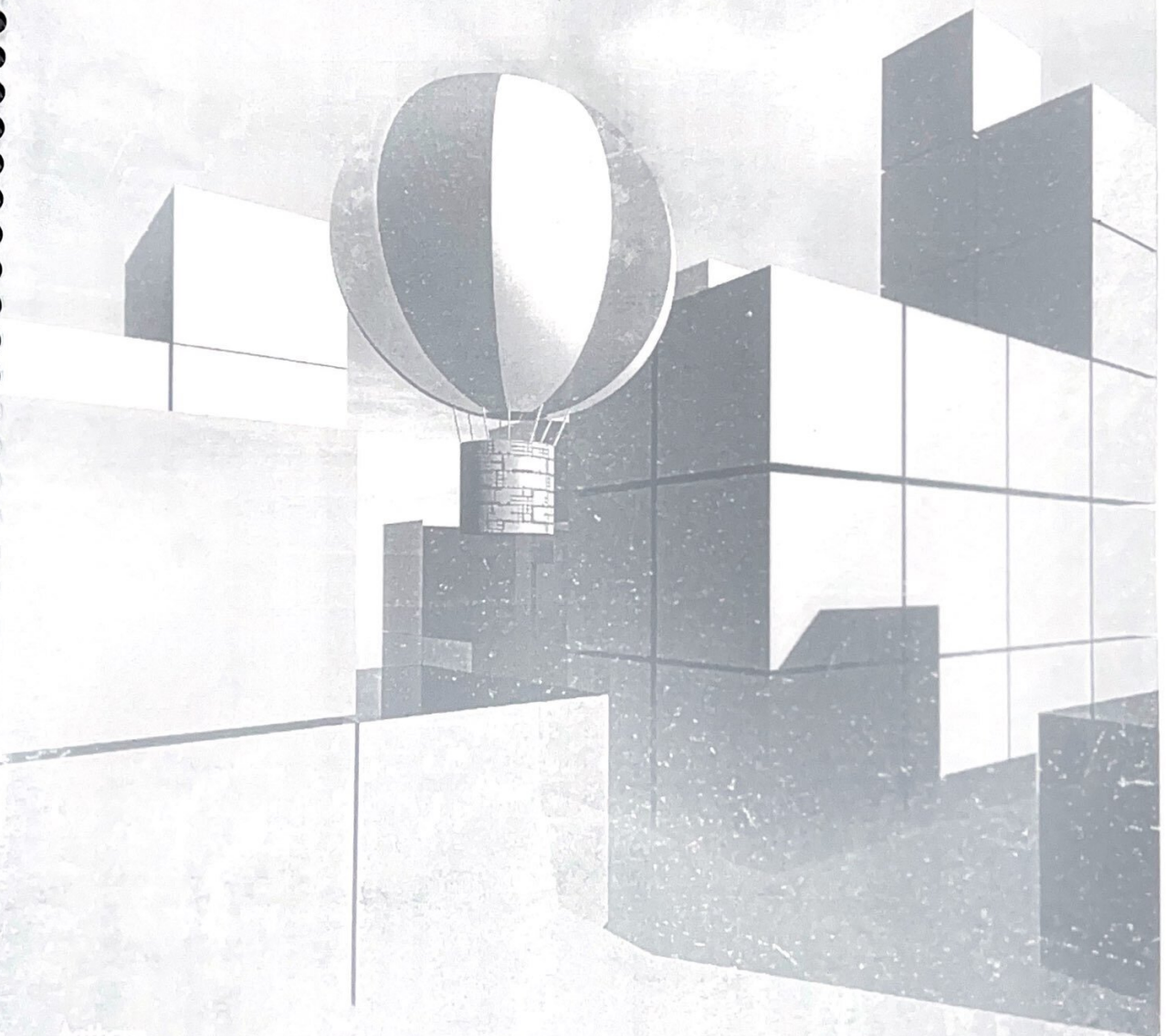
Teacher's Resource Book
for Dr. Super's Factor Blocks™

Authors
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Discovering Greatest Common Factor and Least Common Multiple

With Hands-on Factor Blocks™



Authors
Behrouz Agha
Noel Bernal
Mark Apple

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Mastering the Multiplication Table

with Dr. Super's Factor Blocks

+		X
12=12		1x12=12
6+6=12		2x6=12
4+4+4=12		3x4=12
3+3+3=12		4x3=12
2+2+2+2+2+2=12		6x2=12

Authors
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Mark Spikell
Noel Klimenko
Bonnie Glazewski

The
United
States
of
America



The Commissioner of
Patents and Trademarks

Has received an application for a patent for a new and useful invention. The title and description of the invention are enclosed. The requirements of law have been complied with, and it has been determined that a patent on the invention shall be granted under the law.

Therefore, this

United States Patent

Grants to the person(s) having title to this patent the right to exclude others from making, using, offering for sale, or selling the invention throughout the United States of America or importing the invention into the United States of America for the term set forth below, subject to the payment of maintenance fees as provided by law.

If this application was filed prior to June 8, 1995, the term of this patent is the longer of seventeen years from the date of grant of this patent or twenty years from the earliest effective U.S. filing date of the application, subject to any statutory extension.

If this application was filed on or after June 8, 1995, the term of this patent is twenty years from the U.S. filing date, subject to any statutory extension. If the application contains a specific reference to an earlier filed application or applications under 35 U.S.C. 120, 121 or 365(c), the term of the patent is twenty years from the date on which the earliest application was filed, subject to any statutory extension.

J. Todd Pichini

Acting Commissioner of Patents and Trademarks

Pamela L. Morton

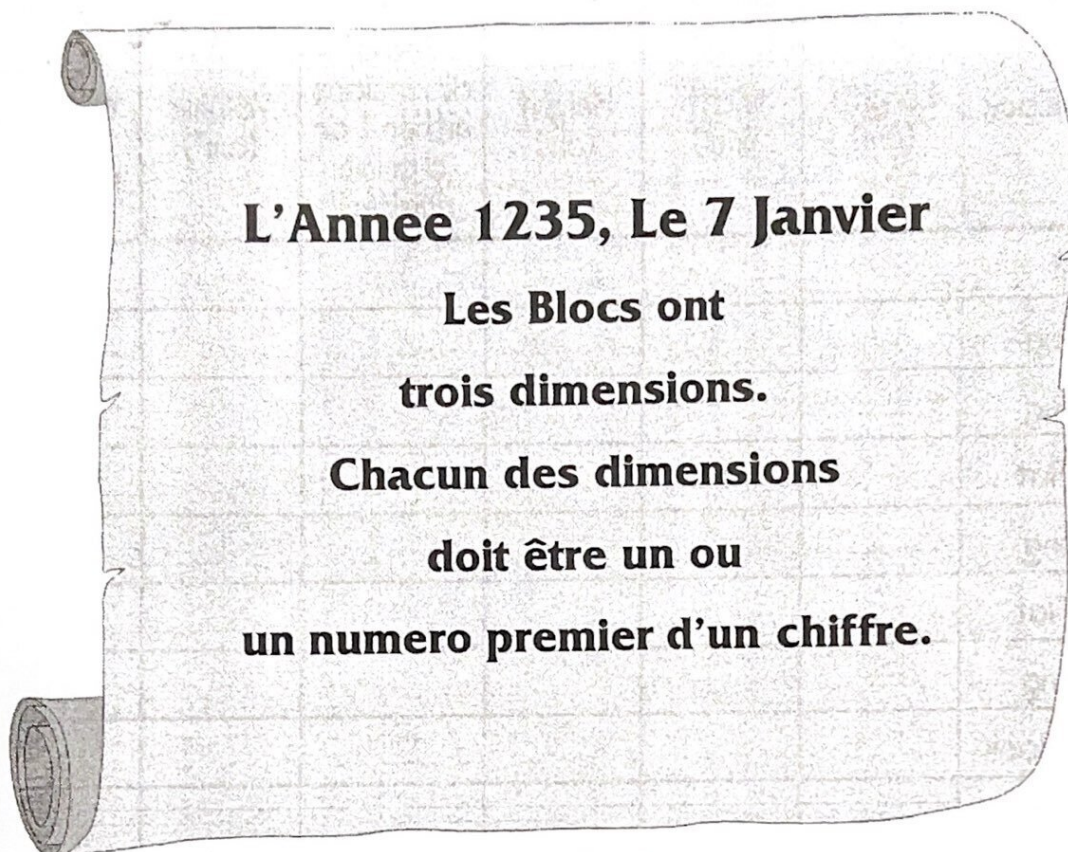
Attest

Activity Sheet 3

Name _____

Date _____

Ancient Scroll



Translation Box

French	English
chacun	each
chiffre	Number
doit	Must
des	the
être	is
les	these
numero	Number
ou	or
premier	Prime
trois	Three
un	one

Complete the Height and find the Name and the Number of Units that are missing from Dr. Super's Factor Blocks set.
Why do you think these were not included in the set?

Type	Height	Width	Length	Name	Number of Units
Block	2	3	3	$2 \times 3 \times 3$	18
Block	3	3	3	$3 \times 3 \times 3$	27
Block	3	3	5	$3 \times 3 \times 5$	45
Block	5	5	5	$5 \times 5 \times 5$	125
Block	2	5	5	$2 \times 5 \times 5$	50
Block	3	5	5	$3 \times 5 \times 5$	75
Block	1	2	7	$1 \times 2 \times 7$	14
Block	2	3	7	$2 \times 3 \times 7$	42
Block	3	5	7	$3 \times 5 \times 7$	105
Block	1	3	7	$1 \times 3 \times 7$	21
Block	2	5	7	$2 \times 5 \times 7$	70
Block	1	5	7	$1 \times 5 \times 7$	35
Block	2	2	7	$2 \times 2 \times 7$	28
Block	3	3	7	$3 \times 3 \times 7$	63
Block	5	5	7	$5 \times 5 \times 7$	175
Block	2	7	7	$2 \times 7 \times 7$	98
Block	3	7	7	$3 \times 7 \times 7$	147
Block	5	7	7	$5 \times 7 \times 7$	245
Block	1	7	7	$1 \times 7 \times 7$	49
Block	7	7	7	$7 \times 7 \times 7$	343

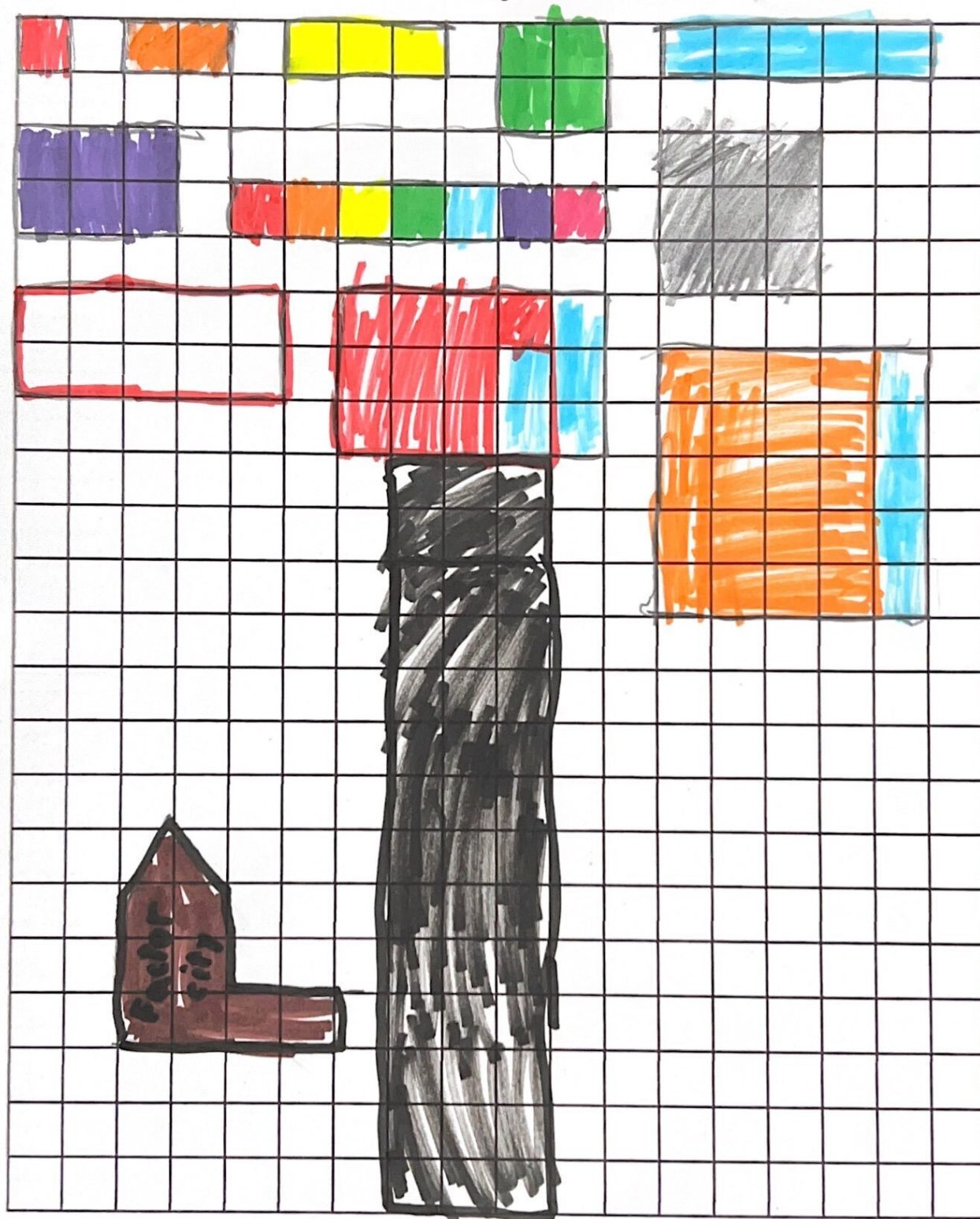
Factor Blocks are rectangular prisms that have 1 or a prime digit for the length of their 3 dimensions.

Activity Sheet 3

Name _____

Date _____

Bird's-Eye View



Activity Sheet 1

Name _____

Date _____

Meet the Factor Blocks™

Factor Blocks	Type	Height	Width	Length	Name	Number of Units
	Unit	1	1	1	1	1
	Long	1	2	1	2	2
	Long	1	3	1	3	3
	Long	1	1	5	5	5
	Long	1	7	1	7	7
	Flat	1	2	2	2×2	4
	Flat	1	2	3	2×3	6
	Flat	1	3	3	3×3	9
	Flat	1	2	5	2×5	10
	Flat	1	3	5	3×5	15
	Flat	1	5	5	5×5	25
	Block	2	2	2	$2 \times 2 \times 2$	8
	Block	2	3	2	$2 \times 2 \times 3$	12
	Block	2	5	5	$2 \times 2 \times 5$	20
	Block	2	3	5	$2 \times 3 \times 5$	30

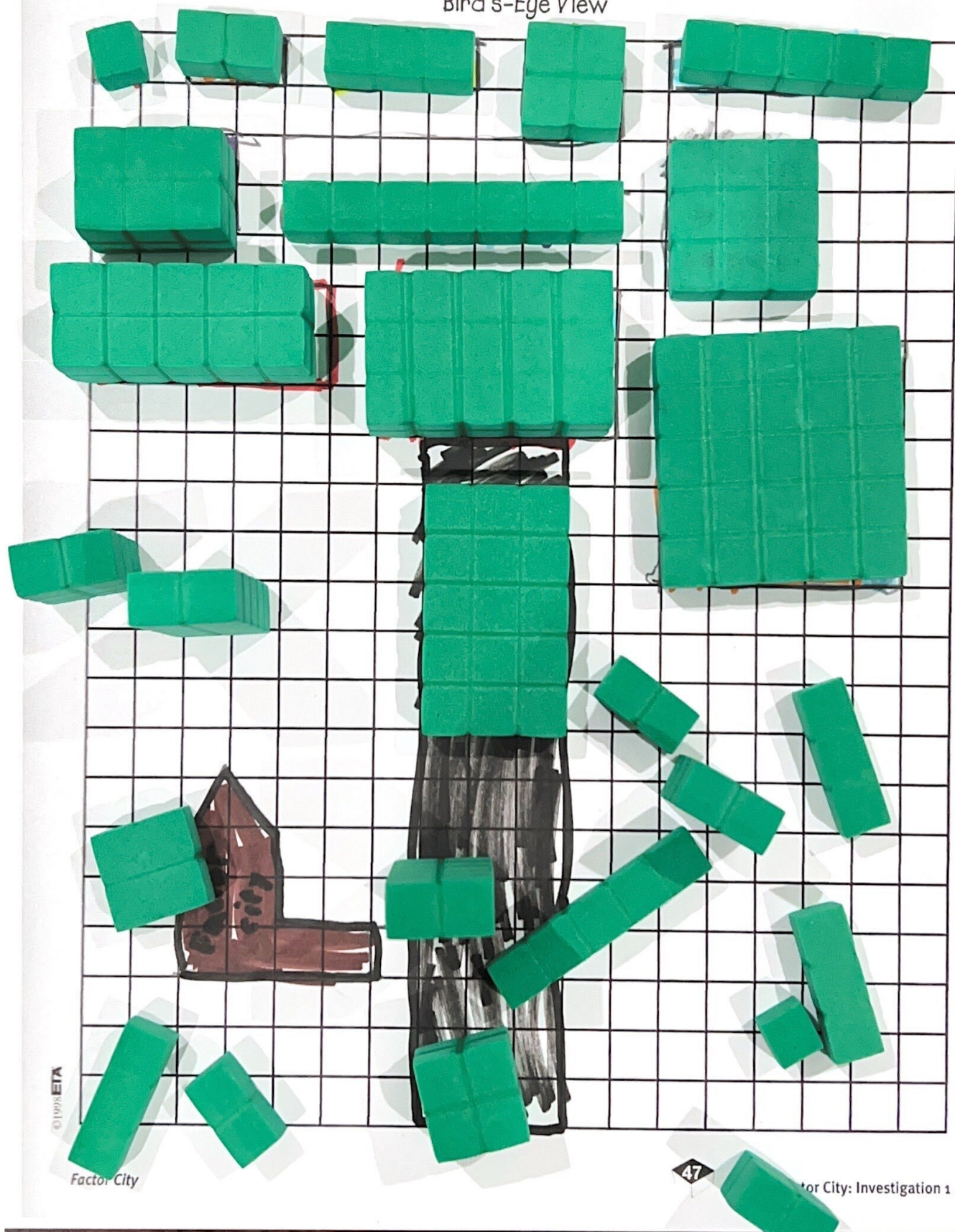
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Activity Sheet 3

Name _____

Date _____

Bird's-Eye View



Math Circle 7 10/5/2022 1:50 to 3:00 pm

Cyrus, Marc and Dar

Before continuing the activities with Factor Blocks we watched 2 videos on YouTube called [Powers of Ten™ \(1977\)](#) made by Eames husband and wife that is amazing not only to give a great introduction to the positive and negative power of 10 but also to give a sense of where we as humans fit into the universe at large and the microscopic world also. We also watched the more recent version of this video called [Scales of the Universe in Powers of Ten](#) which is very similar with some better images when we travel out into the cosmos and also to the microscopic world. They enjoyed it and we did talk about the powers of ten some and I gave them the Conversion Factors for meter, liter and gram. I think this table was a bit overwhelming but Marc paid more attention to it. Cyrus sort of dismissed it. I mentioned that when building Factor City we will discuss Scale and some of this will make more sense. I did do a sheet for Cyrus' morning paper that I will attach where I tried to use color and font to relate powers of ten to the conversion factors with their names. This still turned out to be challenge for Cyrus and there were so many new names but later we discussed it and he was happier with it.

Next they worked on examining how a Factor Blocks can be rotated to fit on different plot using the plots that they had from last session and also how many factor blocks can be placed on the same plot.. Then they completed Activity Sheet 4, Table 1. They did that fast but did not spend enough time to actually place the Factor Blocks. Cyrus did more of the placing of the Blocks than Marc but they wanted to fill up the table and they did a good job.

Next we moved on to Activity Sheet 4, Table 2 where they find on how many different plots each Factor Block can be place. Here they used the blocks more but I had to keep encouraging them to use the blocks before they filled the table. We did have a good discussion about the number of faces that each block has and how this related to the table they had filled up.

They wanted to stop so we will do the activity for relating floors to a building to the area of the plot and finding multiple to be done before the

next session at home.

They also got the nets for the building for Factor City that they will color and cut and make into rectangular prisms with tape (either at home in in the next session on 10/19/2022).

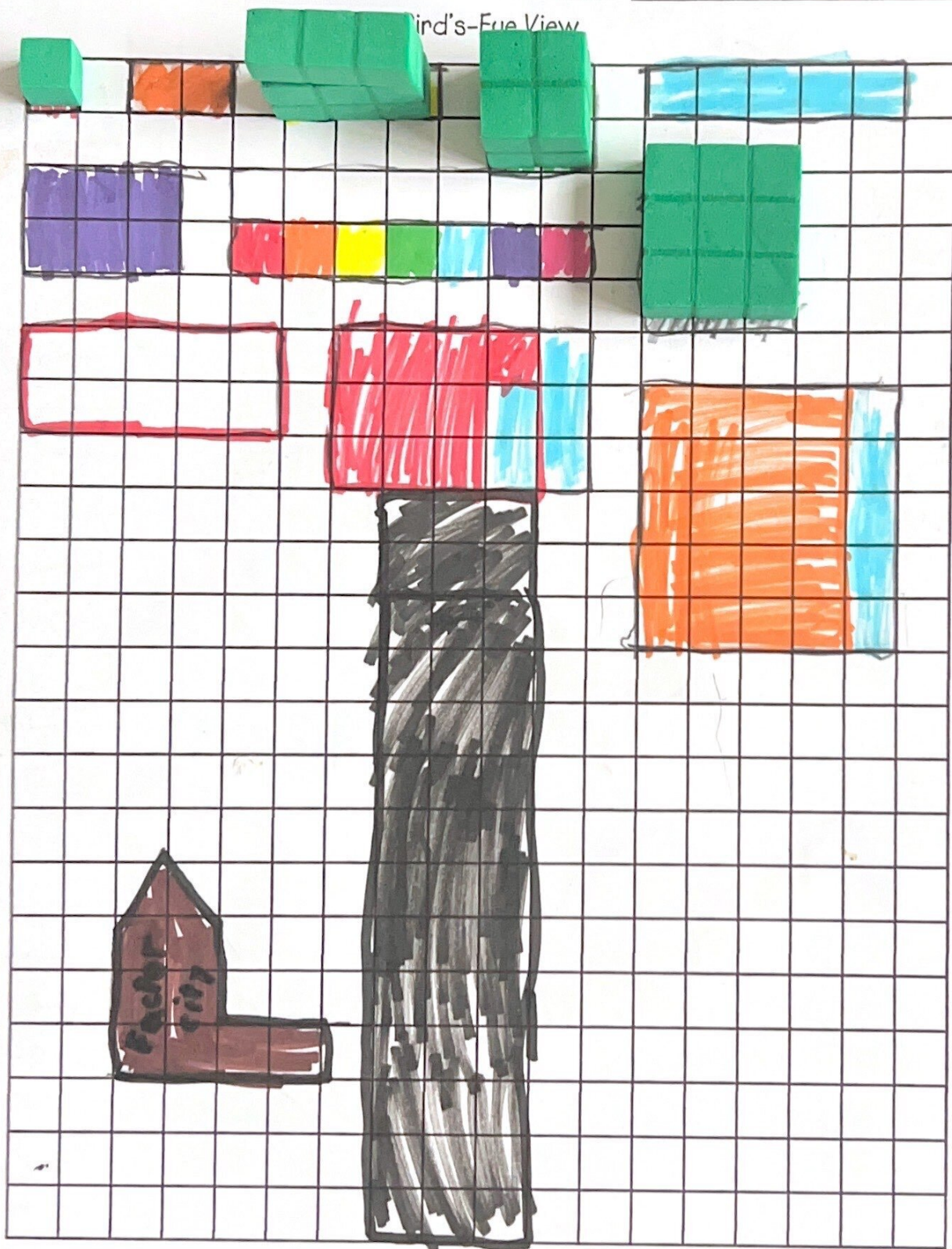
Nearly half of our session was taken up by the Powers of Ten Videos but I think this was well worth the time and you can have them watch these again.

Activity Sheet 3

Name _____

Date _____

Bird's-Eye View



Activity Sheet 4

Name Cyrus

Date 10.5.20

Table 1

Plot Area

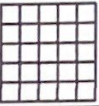
Plot Drawing	Plot Dimension	Plot Area	Factor Blocks Buildings that Fit on this Plot
	1x1	1	1, 2, 3, 5, 7
	1x2	2	2, 4, 6, 10
	1x3	3	3, 6, 9, 15
	1x5	5	5, 10, 15, 20
	2x2	4	4, 8, 12, 20
	2x3	6	6, 12, 30
	3x3	9	9
	2x5	10	10, 20, 30
	3x5	15	15
	5x5	25	25

Table 2

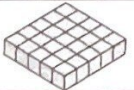
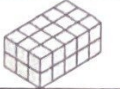
Factor Blocks that only Fit on 1 Plot (all faces the same)	Factor Blocks that Fit on Exactly 2 Plots (2 different faces)	Factor Blocks that Fit on Exactly 3 Plots (3 different faces)	Factor Blocks that Fit on Exactly 4 Plots (4 different faces)
Unit	2x2 Flat	1x2x3	0
2x2x2 "8"	3x2	1x2x5	
	1x3	1x3x5	
	2x2	2x3x5	
	1x5		
	1x7		
	3x3		
	2x2x3		

Activity Sheet 1

Name _____

Date _____

Meet the Factor Blocks™

Factor Blocks	Type	Height	Width	Length	Name	Number of Units
	Unit	1	1	1	1	1
	Long	1	2	1	2	2
	Long	1	3	1	3	3
	Long	1	1	5	5	5
	Long	1	7	1	7	7
	Flat	1	2	2	2×2	4
	Flat	1	2	3	2×3	6
	Flat	1	3	3	3×3	9
	Flat	1	2	5	2×5	10
	Flat	1	3	5	3×5	15
	Flat	1	5	5	5×5	25
	Block	2	2	2	$2 \times 2 \times 2$	8
	Block	2	3	2	$2 \times 2 \times 3$	12
	Block	2	5	5	$2 \times 2 \times 5$	20
	Block	2	3	5	$2 \times 3 \times 5$	30

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Math Circle 8 10/19/2022 1:50 to 3:45 pm

Cyrus, Marc and Dar

10/12/2022 was cancelled due to fall break

Marc and Cyrus completed the Factor City Activity and it was a very long session with no breaks. They colored taped and, sang some, making their cities. I was amazed that they lasted almost 2 hours without complaining. good for them. (This activity is really for 4-6 graders so it requires more dexterity for cutting and making the buildings)

First they initialed and colored the different buildings on Activity Sheet 12 using the buildings they had cut out and glued together. They had both colored the building before and Mark had cut his, For Cyrus I did the cutting before the session and had glued his houses and also glued Marc's when they worked with Activity sheet 12. This was a slow process but they enjoyed it. I encouraged Marc to color more carefully and Cyrus to work a little faster. Marc likes to go very fast and Cyrus goes real slow. It is good for both of them to adjust a little towards the middle speed.

Next they completed Activity Sheet 12 and chose and colored the plots where they would place each building and used the same color as the corresponding buildings and initialed them. Then they built the houses with the Factor Blocks that were designated on the designated plots. Marc took the initiative to build the houses very fast then I worked with him to use the Blocks that were designated. Cyrus took his time and used a lot of time to build all the houses with the Factor Blocks that were indicated.

Next I worked with both of them to put in the buildings that they had built on their corresponding plots with double-sided glue. They enjoyed this process and blew on the their own houses as well as on their partners to see if they fly off or not. This was not as slow as the other activities but proceeded at a good rate for both of them.

At this point we were nearly an hour and a half into this session and we still had the Activity Sheet 13 to complete to find the total cost for the city. I asked them if they wanted to complete it now or take it home and Marc said he had to do it now as he had Soccer later. Both of them really wanted to complete their work. As usual Marc went really fast and Cyrus really slow. I worked to adjust their speeds a little and in particular encouraged Marc to write a little slower and more legibly <he did say that he did not think that was important and I insisted that it was;=> Figuring out the Total Cost was not easy but I pointed out that they could look at Activity Sheet 12

and there was a clue there. They did could figure out the Total Cost when they realized that the answer was the total value multiplied by the \$10,000 which is the Price of a Unit volume.

I did forgo the discussion of Scale (I will talk about it next session briefly) and also the 3-D drawing of the city (I think this is too complicated for them at this stage but I will have them do some 3-D Factor Blocks where I have already traced them and marked the corners).

These guys are amazing! You should all be proud of them and let them know that Dar (Dr. Super) said so;=>

Activity Sheet 13

Name _____

Date _____

Factor City Budget Chart (Accountant)

Rate: \$10,000 per 1cm^3

Building	Factor Blocks Used	Number of Factor Blocks Used	Value of Each Factor Block Used	Building Cost
Gas Station	Unit	6	10,000	60,000
House	2 Long	6	20,000	120,000
Video Store	3 Long	4	30,000	120,000
Public Library	5 Long	2	50,000	100,000
Warehouse	7 Long	1	70,000	70,000
Grocery Store	2x2 Flat	3	40,000	120,000
Movie Theater	2x3 Flat	2	60,000	120,000
Stadium	2x5 Flat	2	10,000	20,000
Restaurant	3x3 Flat	1	90,000	90,000
Post Office	3x5 Flat	1	150,000	150,000
Supermarket	5x5 Flat	1	250,000	250,000
Ice Cream Parlor	2x2x2 Block	1	80,000	80,000
Hospital	2x2x3 Block	1	120,000	120,000
School	2x2x5 Block	1	200,000	200,000
Office Building	2x3x5 Block	1	300,000	300,000
Totals		33		2,100,000

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Activity Sheet 12

Name _____

Date _____

Factor City Building Chart (City Planner)

1 Unit (1x1x1 Factor Block) = 1cm³

	Building	Factor Blocks Used	Number of Factor Blocks Used	Volume (cm ³)
GS	Gas Station	Unit	6	6
H	House	2 Long	6	12
VS	Video Store	3 Long	4	12
PL	Public Library	5 Long	2	10
W	Warehouse	1 Long	1	7
GS	Grocery Store	2x2 Flat	3	12
MT	Movie Theater	2x3 Flat	2	12
ST	Stadium	2x5 Flat	2	20
R	Restaurant	3x3 Flat	1	9
PO	Post Office	3x5 Flat	1	15
Su	Supermarket	5x5 Flat	1	25
ICP	Ice Cream Parlor	2x2x2 Block	1	8
H	Hospital	2x2x3 Block	1	12
SC	School	2x2x5 Block	1	20
OB	Office Building	2x3x5 Block	1	30
	Totals		33	210

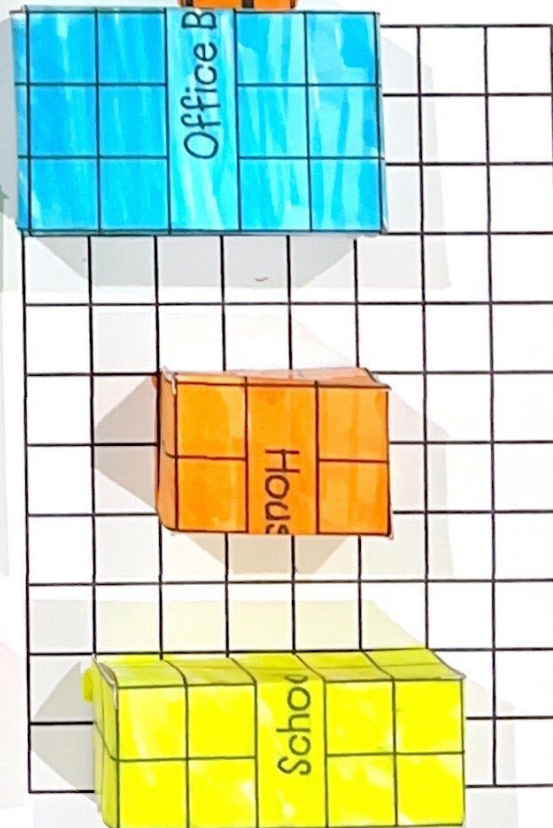
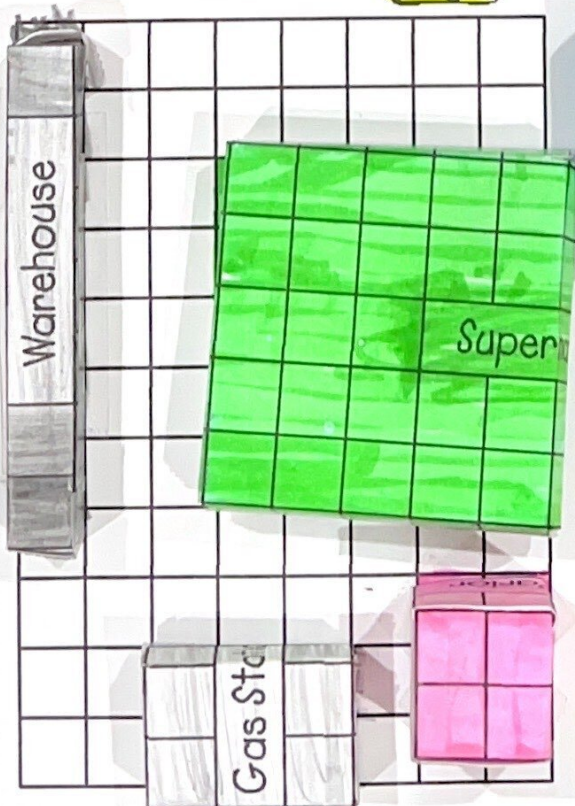
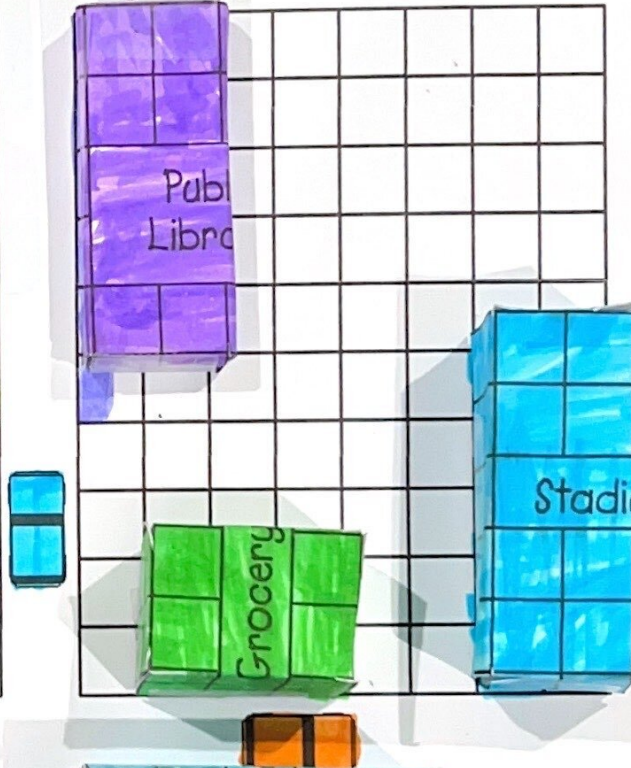
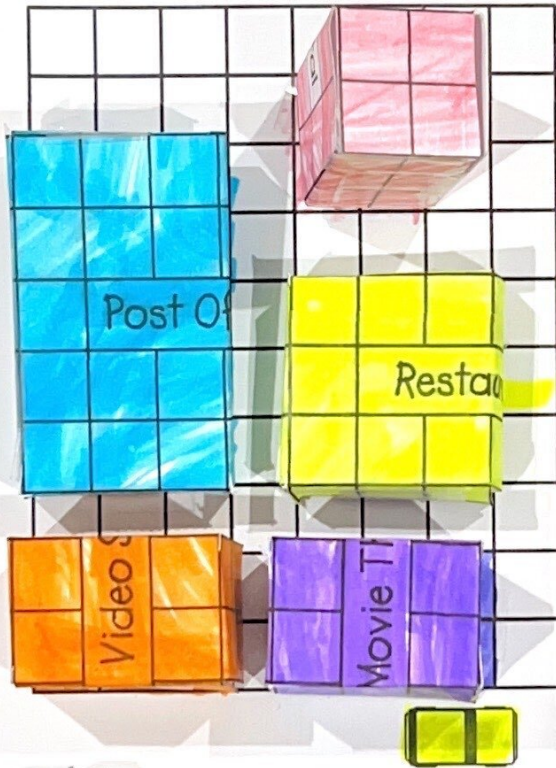
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Activity Sheet 14

Name _____

Date _____

Bird's-Eye View (Engineer)



Math Circle 9 10/26/2022 12:30 to 2:45 pm

Cyrus, Marc and Dar

We played the game of "Divisibility Rules!" that I had invented for them.

Rules for the game "**Divisibility Rules!**"

The rules of The game are somewhat like Monopoly in that each player gets \$500 and you go around a board by rolling two dice and try to have the most money at the end of the game.

Each of the squares has a number from 1 to 20 consecutively. When you fall on a square you get a small card with a 3-digit number and need to see what number it is divisible by. You will get 10 times the amount of the divisor if you guess correctly and the card will be placed on that number (There are 3 cards for each square). If you happen to have a card that is divisible by the number on the square that you have landed on you will get 5 times the normal amount. So if you get a card on square 20 that is divisible by 20 you will get \$1,000!

At The beginning of the game each players gets a "Calculator Check" and a "Calculator Challenge" card and each time they pass the Big/Divisor (Starting position) they get one more card with a maximum of 2 cards of each kind. The player will also get an extra \$50 each time they pass the Big/Divisor. The calculator cards can be used for a divisor that a player is not sure of and the Calculator Challenge to challenge an opponent. If the Challenge is correct the losing person will give the other player half of what they got from the Bank and place half in the middle. The person that lands on "Win the Bank" corner square will get this money.

There are also two other corner Squares that will make a player pay \$100 to the middle or lose their turn.

Players will always have a chance to move a card that is placed too low to a bigger divisor. For example if 484 is placed on 4 the next player or the same player on their next turn can move it to 8 or 16 and get \$80 or \$160. Cards can be moved many times. At the end of the game there will be more than 3 cards on 1 and 2 and players should attempt to place them on the proper higher divisor before they count their money.

All the divisibility rules are on cards that are on the board and can be consulted by a player at any time. In addition all the values for multiples of

7, 13, 17, and 19 up to 100 were displayed on the board. Eventually the players will remember these as part of the multiplication table.

The game ends when all the 60 number cards have been placed on the squares. This took about 2 hours so maybe the number of cards should be limited to 40.



Marc and Cyrus decided to play as a team. This was good as Cyrus had more of an introduction to some of the rules in particular for 7, 11, 13, 17 and 19 so they were only competing with me. The game took more than 2 hours and they were having a great time. I did a few challenges were they won and also they did challenges that were successful some by mistake and some on purpose to make it exciting. They did take their time and discussed how and why a number was or was not divisible by something. Calculator Check was not used as much as I thought it would be.

Even though the goal for the kids was to get as much money as possible but at each turn for themselves and for me essentially all the rules were reviewed and that is why the game took so long.

If I had the energy and also had not seen how education games are at the mercy of the publishers I would have tried to patent and market this game.

I think it would be very successful for kid in grades 4-8. In particular if we make a few modifications and add the explanations for where the rules for 7, 11, 13, 17, and 19 come from which we will attempt in the next session.

I gave them a sheet with all the rules up to 20 and am working on one up to 100. Even though knowing these divisibility rules is not that important mathematically but similar to mental double digit multiplication gives a lot of confidence to the kids as most grown-ups can not perform these tasks.

All in all this was the most successful activity for Math Circle and I think Mar and Cyrus were both delighted. Unfortunately I did not take a picture with both of them and took this one later when Cyrus wanted to play again before we left for Sedona the next day.



Divisibility Rules from 1 to 20

7, 12, 17, 19 on the Back

1	Everything
2	Even - 1s-digit=0,2,4,6,8
3	Sum of Digits Divisible by 3
4	First two digits divisible by 4 O2,O6 or E0,E4,E8
5	1s-digit= 0 or5
6	Divisible by 2 and 3
8	The last 3 digits divisible by 8
9	Sum of Digits Divisible by 9
10	1s-digit= 0
11	Alt Sum Divisible by 11
12	Divisible by 3 and 4
14	Even and divisible by 7
15	Divisible by 3 and 5
16	The last 4 digits divisible by 16
18	Even and Divisible by 9
20	1s-digit=0 and even 10s-digit

Divisibility Rules from 1 to 20

7, 13, 17, 19 on the Back

For 7, 13, 17 and 19

Write your Number **ab** as $ax10 + b$

e.g. $791 = 790 + 1 = 79 \times 10 + 1$

7 ab is divisible by 7 iff $a-2b$ is divisible by 7

e.g. 252 is divisible by 7 because $25 - 4 = 21$ that is divisible by 7

13 ab is divisible by 13 iff $a+4b$ is divisible by 13

e.g. 533 is divisible by 13 because $53+12 = 65$ that is divisible by 13

17 ab is divisible by 17 iff $a-5b$ is divisible by 17

e.g. 731 is divisible by 17 because $73-5 = 68$ that is divisible by 17

19 ab is divisible by 19 iff $a+2b$ is divisible by 19

e.g. 437 is divisible by 19 because $43+14=57$ that is divisible by 19

Math Circle 10 11/2/2022 12:00 to 12:45 1:15 to 2:15 pm

Cyrus, Marc and Dar

To start the session I gave them a challenge. They were asked to calculate the number of unit cubes in a one quarter pyramid that Rooz had built some time back (see picture). The base is a 10x10 square coming to 1. So they had to calculate the sum of squares from 1 to 10. I left the room and came back in about 5 minutes. They announced that they used different methods and came up with the same answer 385. I had done the same using the formula for the sum of squares $(n)(n+1)(2n+1)/6$ and had a different answer (incorrectly). I was surprised how they both got the same wrong answer. So I showed them the formula and asked them to do it with a little help from me in simplifying the fraction. They got both 385 again! It was great.



Then I gave them 2 different sheet with 2 numbers divisible by 11 for Marc and 7 for Cyrus (randomly) (see picture) and asked them what number they were divisible by and they correctly said 11 and 7 (Cyrus said also 3 which was their inadvertently) then I asked them to add and subtract their 2 numbers. And see if the two results were still divisible by the same number. They did it and quickly said of course yes.

Next I gave them 12 cards (see picture) and asked them to find the right cards for their page that had 6 places to put the cards and helped them place their cards. This took a little bit in particular for 7 that has twist. The idea is to give them a proof for the divisibility rules which is as follows.

For 11 that we did first together a number ab ($10a+b$ where b is the 1-s digit)

ab is divisible by 11 iff $a-b$ is divisible by 11 (this I had not seen anywhere else before!)

The reason is

$$\begin{array}{r} 10a+b \\ + \quad a-b \\ \hline \end{array}$$

$11a$ that is always divisible by 11

Next they proved the Cyrus' numbers that were divisible by 7. Now they had to multiply by a number before they added the original number and the for 7 for ab the rule is $a-2b$.

$$\begin{array}{r} 10a + b \\ +4(a-2b) \\ \hline \end{array}$$

$14a - 7b$ That is divisible by 7

They had a lot more trouble with this and we need to talk about it later again.

Next They had their Lunch.

We then played the game and again Marc and Cyrus teamed up to beat Dr. Super and they did! This time Marc also rolled the dice and did all his own numbers. We all would look at the number and suggest possibilities and they game was much more collaborative than competitive.

Unfortunately Laura showed up with the Marc's Pokémon cards at 2:15 and they abandoned the game to do their Pokémon exchange;=>

This game was a great success and trying to explain why has given them a taste of pre-algebra. I am very happy as I think they really have enjoyed this game;=>

math



100

81

64

49

36

25

16

9

4

1

7

45

240

+ 100

385

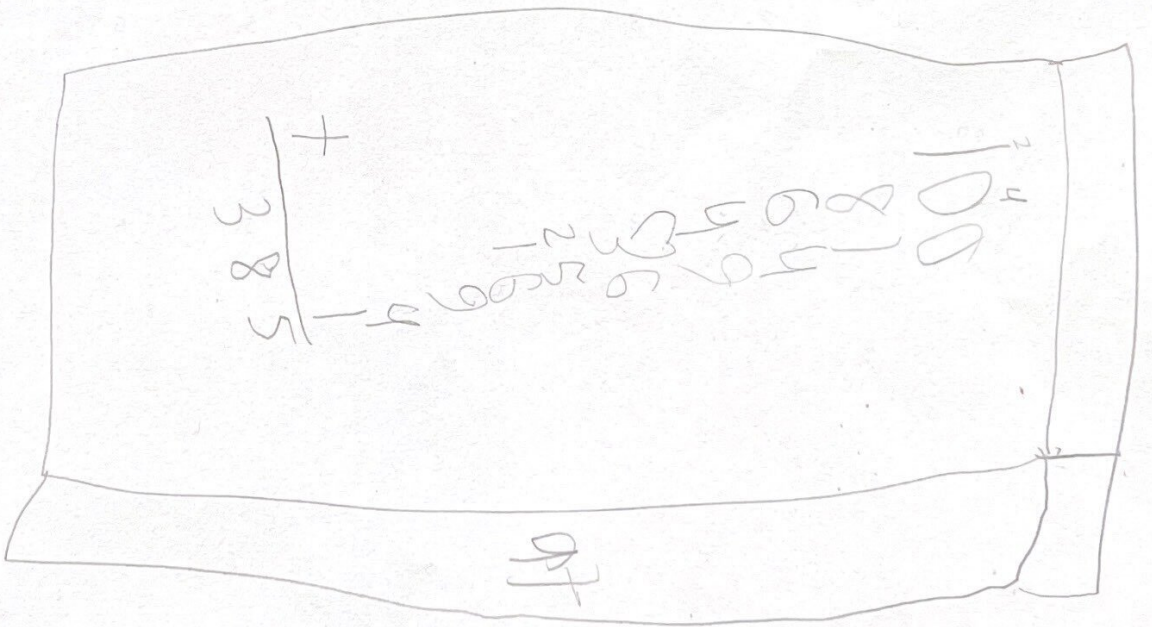
$$\begin{array}{r} 5 \quad 7 \\ \cancel{10} \times \cancel{11} \times \cancel{21} = 6 \end{array}$$

221

A+

WOW





$$\frac{10 \times 11 = 110}{11 \times 11 = 121}$$

38

77

22

11

$$\begin{array}{r} 77 \\ 22 \\ \hline 99 \end{array}$$

$$\begin{array}{r} 77 \\ -22 \\ \hline 55 \end{array}$$

21

63

7 & 3

$$\begin{array}{r} 21 \\ + 63 \\ \hline 84 \end{array}$$

$$\begin{array}{r} 63 \\ - 21 \\ \hline 42 \end{array}$$

$a \ b$

161

$4(a-2b)$

56

+

217

$10a+b$

$10 \times 16 + 1$

$4(a-2b)$

$4 \times (14)$
 $4 \times (16-2)$
 $4 \times 16 - 8$

+

$14 \times 16 - 7$

ab

352

$a-b$

+

33

385

$10a+b$

$10 \times 35 + 2$

$a-b$

+

35-2

11×35

Math Circle 11 11/15/2022 2:00 to 3:20 pm

Cyrus, Marc and Dar

We played a new game called “Divisibility Graphs!” that I had invented for them. This is based on David Wilsons [Divisibility Graphs extended and simplified and explained by James Tanton.](#)

Rules for the game “**Divisibility Graphs!**”

The rules of this game are also somewhat like Monopoly in that each player gets \$300 and you go around a board by rolling two dice and try to have the most money at the end of the game.

Each of the squares has a number from 1 to 12 and the Divisibility Graph for that number.

Players will start on Square 1 and go around the board by rolling 2 dice. When you fall on a square you roll over 3 number cards and make a 3 digit number. There are 3 sets of cards from 0 to 9. Players discard the 3 used cards and when the pile is finished they get reshuffled and reused.

Then the player will move to the board with the number that he think will divide his number and go around it using the digits of his number as described below. If you come back to 0 You will get 10 times the number on that board in dollars. if not you will lose your turn. If you can go around the board that you have landed on you will get 50 times that number in dollars and what is in the Bank.

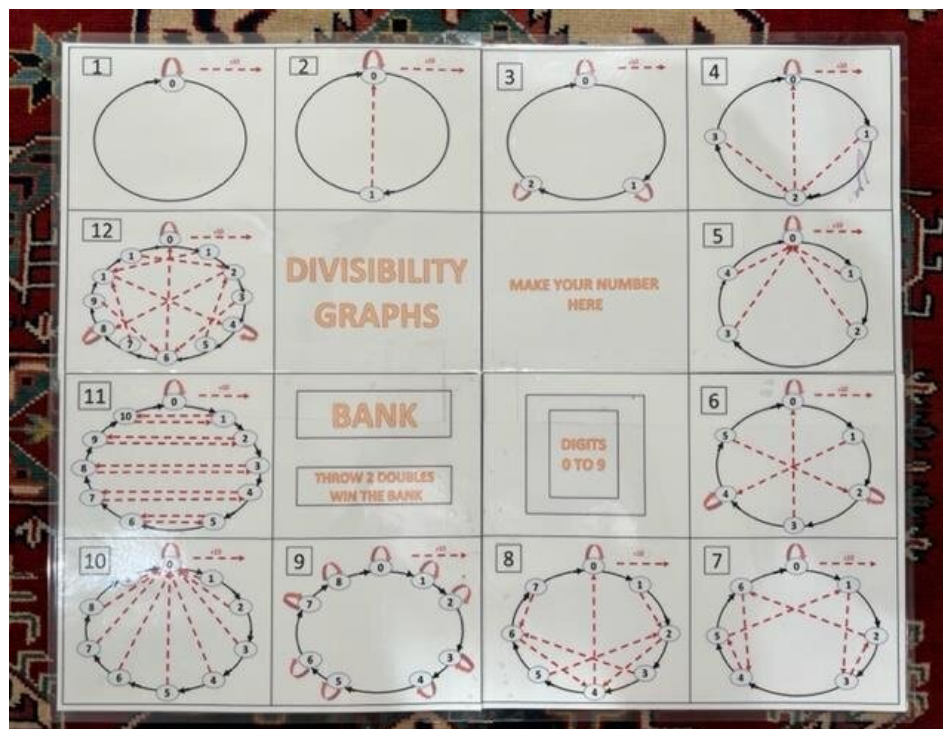
On the board you start with the 100s digit and go around the divisibility graph that is selected that many steps. Then you follow the red line on the board. Next you do the same for 10s digit and last for the units. On the last move you will not use the red line (if you on zero correctly it does not matter). If you end up on another number that is the remainder of dividing your number by the number on the graph.

You will get another roll when you roll a double.

When you come back to square 1 or pass over it because of the dice roll or going to a board to test the divisibility you get \$50 and \$50 goes in the Bank. Any player who rolls doubles twice in a row will get all the money in the Bank.

Each time you complete a board you will get a token for that number. When a player has collected all 12 tokens the game will end if he declares “Money

Count” and all the players will count their money and the player with most money will win the game. The player who has a complete set of Tokens can declare “Money Count” when his turn comes or wait to make more money before he does that.



We started with a challenge again. I gave them the same problem with the sums of the squares but this time going to 50. Cyrus said we can use the result from last time 385 and add in the rest. I said that you could but it will be easier to use the formula $n(n+1)(2n+1)/6$ for the sums of squares that they had seen before and was on the board and the calculator. I left and came back and they had both figured out the answer that was 42,925 correctly.

Then I introduced (Cyrus really did as I had shown him the game) the Divisibility Graphs Game. And we started playing. They wanted to be partners again but I insisted that they should both play. And they did. The really interact well and enjoy playing against me and not competing together. They particularly like beating me;=>

We played for early 1:30 hours and Rooz joined in and was watching for part of the game and kept their Tokens and money.

The most important part of the game is for them to manipulate the 3 digits to make it divisible by the optimum number or to get a token that they did not have. They took their time and were really engaged on this.

I think they are both curious about why going around a board will determine divisibility that we shall discuss in the next session after the break.

I had also figured out all the divisibility rules up to 100 and made a small card that I gave them that they consulted for 4 and 8 in particular. They made no mistakes and I made a couple;=> that delighted them. They completed their Tokens first and I was missing 1 and they won by about \$500 it was nearly \$3500 to \$3000.

All in all an excellent session with no breaks and we had a lot of fun. I did ask them which game they preferred Divisibility Rules or Divisibility Graphs and they both said Divisibility Rules!

I did make a board and numbers for Marc to take home and try out. And I think it will be great for Hugo to figure out where the rules come from and the why the divisibility graphs work.

I was so excited playing with them that I forgot again to take a picture or short Video while we were playing. That is a shame.

Math Circle 12 11/30/2022 1:50 to 3:00 pm

Cyrus, Marc and Dar

We started with a challenge again. I gave them a similar problem. They were to find the total number of balls in a triangular pyramid with base 8. I drew a really bad picture (and I think this confused them some). I gave them a few minutes to work on it. When I came back they had different answers both relatively large. I did explain that they could see that we had different layers that were the sums of consecutive numbers first to 8 then to 7. Cyrus came up with the answer then that was 120 by adding these.

I showed them the formula that I had come up with which is $2n(n+1)(2n+1)/3$ if $n=N/2$ when N is even (in this case it is 8) and is equal to $(n+1)(2n+1)(2n+3)/3$ when $N=2n+1$. I tried to explain how I had found this but it got too complicated. They did however check that this would work for $N=8$ as $2 \times 4 \times 5 \times 9 / 3 = 120$.

Then we played the Divisibility Rules number cards from the Divisibility Graphs game. They wanted to be partners again. This made the game more challenging and interesting as they could manipulate the 3 digits to get a number that had the largest divisor or the divisor they were. We went thru the cards 3 times until 3:00. I asked them if they wanted to go another round but Mark said we should stop and Cyrus agreed. Rooz also came for a while and watched and also did other things in the room. Then we counted our money and they won again!

I did not get to explain the reason the Divisibility Graphs work but may do it in a later session.

I had revised the divisibility cards to 100 and gave them the new ones. Now almost all of these can be done mentally.

Hugo also came with Laura and I got to meet him. I did show him the 100 cards and challenged him to work on where these rules came from. I also or the book "A Short History of Everything" and we agreed that he may work on it later.

It was again a great session with no breaks and we had a lot of fun and good math.

3 Digits Sum Divisible by 3	7 a-2b Divisible by 7	3 Digits Sum Divisible by 3	7 a-2b Divisible by 7
9 Digits Sum Divisible by 9	13 a+4b Divisible by 13	9 Digits Sum Divisible by 9	13 a+4b Divisible by 13
11 Alt Digits Sum Divisible by 11	17 a-5b Divisible by 17	11 Alt Digits Sum Divisible by 11	17 a-5b Divisible by 17
4 2a-3b Divisible by 4 (2 Digits)	27 a-8b Divisible by 27	4 2a-3b Divisible by 4 (2 Digits)	27 a-8b Divisible by 27
8 2a-3b Divisible by 8 (3 Digits)	23 3a-2b Divisible by 23	8 2a-3b Divisible by 8 (3 Digits)	23 3a-2b Divisible by 23
16 2a-3b Divisible by 16	19 a+2b Divisible by 19	16 2a-3b Divisible by 16	19 a+2b Divisible by 19
32 2a-3b Divisible by 32	29 a+3b Divisible by 29	32 2a-3b Divisible by 32	29 a+3b Divisible by 29
64 0,4,8, 2a-3b or 2,6 4a-6b Divisible by 64	49 a+5b Divisible by 49	64 0,4,8, 2a-3b or 2,6 4a-6b Divisible by 64	49 a+5b Divisible by 49
43 3a-4b Divisible by 43	59 a+6b Divisible by 59	43 3a-4b Divisible by 43	59 a+6b Divisible by 59
53 3a-5b Divisible by 53	79 a+8b Divisible by 79	53 3a-5b Divisible by 53	79 a+8b Divisible by 79
73 3a-7b Divisible by 73	89 a+9b Divisible by 89	73 3a-7b Divisible by 73	89 a+9b Divisible by 89
83 3a-8b Divisible by 83	31 a-3b Divisible by 31	83 3a-8b Divisible by 83	31 a-3b Divisible by 31
37 3a+4b Divisible by 37	41 a-4b Divisible by 41	37 3a+4b Divisible by 37	41 a-4b Divisible by 41
47 3a+5b Divisible by 47	61 a-6b Divisible by 61	47 3a+5b Divisible by 47	61 a-6b Divisible by 61
67 3a+7b Divisible by 67	71 a-7b Divisible by 71	67 3a+7b Divisible by 67	71 a-7b Divisible by 71
97 3a+10b Divisible by 97	81 a-8b Divisible by 81	97 3a+10b Divisible by 97	81 a-8b Divisible by 81
3 Digits Sum Divisible by 3	7 a-2b Divisible by 7	3 Digits Sum Divisible by 3	7 a-2b Divisible by 7
9 Digits Sum Divisible by 9	13 a+4b Divisible by 13	9 Digits Sum Divisible by 9	13 a+4b Divisible by 13
11 Alt Digits Sum Divisible by 11	17 a-5b Divisible by 17	11 Alt Digits Sum Divisible by 11	17 a-5b Divisible by 17
4 2a-3b Divisible by 4 (2 Digits)	27 a-8b Divisible by 27	4 2a-3b Divisible by 4 (2 Digits)	27 a-8b Divisible by 27
8 2a-3b Divisible by 8 (3 Digits)	23 3a-2b Divisible by 23	8 2a-3b Divisible by 8 (3 Digits)	23 3a-2b Divisible by 23
16 2a-3b Divisible by 16	19 a+2b Divisible by 19	16 2a-3b Divisible by 16	19 a+2b Divisible by 19
32 2a-3b Divisible by 32	29 a+3b Divisible by 29	32 2a-3b Divisible by 32	29 a+3b Divisible by 29
64 0,4,8, 2a-3b or 2,6 4a-6b Divisible by 64	49 a+5b Divisible by 49	64 0,4,8, 2a-3b or 2,6 4a-6b Divisible by 64	49 a+5b Divisible by 49
43 3a-4b Divisible by 43	59 a+6b Divisible by 59	43 3a-4b Divisible by 43	59 a+6b Divisible by 59
53 3a-5b Divisible by 53	79 a+8b Divisible by 79	53 3a-5b Divisible by 53	79 a+8b Divisible by 79
73 3a-7b Divisible by 73	89 a+9b Divisible by 89	73 3a-7b Divisible by 73	89 a+9b Divisible by 89
83 3a-8b Divisible by 83	31 a-3b Divisible by 31	83 3a-8b Divisible by 83	31 a-3b Divisible by 31
37 3a+4b Divisible by 37	41 a-4b Divisible by 41	37 3a+4b Divisible by 37	41 a-4b Divisible by 41
47 3a+5b Divisible by 47	61 a-6b Divisible by 61	47 3a+5b Divisible by 47	61 a-6b Divisible by 61
67 3a+7b Divisible by 67	71 a-7b Divisible by 71	67 3a+7b Divisible by 67	71 a-7b Divisible by 71
97 3a+10b Divisible by 97	81 a-8b Divisible by 81	97 3a+10b Divisible by 97	81 a-8b Divisible by 81

We started with a challenge again. This time it was related to the Divisibility Rules final and color coded card that I gave them. This card is attached in a separate PDF. They were to find the patterns in particular for the set of rules in blue (numbers ending in 3 and 7) and the ones ending in gray/black (numbers ending in 1 or 9). We had covered the green and the reds ones mostly already and the purple (powers of 2) are pretty much identical. They wanted to start explaining right away but I told them to discuss it and then each of them could choose one of the two colors and explain the pattern. I also told them that my challenge will be to try to explain why these rule and patterns are there!

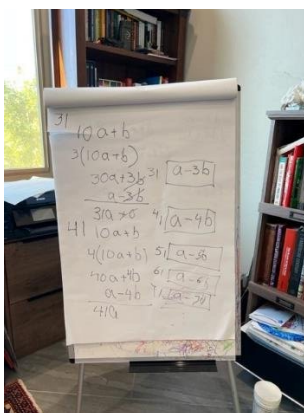
After about 5 minutes I came back. First Cyrus explained (with some help from me) the gray/black and started with rule for 19 that we had used before and the rule for it is $a+2b$ and how the rule for all the rest was a plus the tens digit plus one times b so that for 29 the rule is $a+3b$. He went over all of them and we also talked about 39 and 69 that are missing because they are divisible by 3 and so each one is divisible by 3 & 13 and 3 & 23 respectively and the rule for 13 is already $a+4b$ which would have been the rule for 39. For 69 we could also do $a+7b$ using the pattern. He also explained the rule for numbers ending in 1 and how they all had the rule a minus the tens digit times b . so the rule for 31 is $a-3b$. The rule for 21 is divisibility by 3 and 7 and for 7 we already have the rule $a-2b$ so it would work for 21 also. He went over all for them and we also talked about 51 that is divisible by 3 and 17 and the rule for 17 we already was covered and is $a-5b$ consistent with the pattern.

Then Mark explained the patterns for numbers ending in 3 and 7 (with some help from me) and how the rules all started with $3a$ and then for 3 we had to subtract the tens digit times b and for 7 we had to add the tens digit plus 1 times b . For example for 43 the rule will be $3a-4b$ and for 37 the rule will $3a+4b$. We discussed the numbers we had skipped again. For 23 $3a-2b$ will work and it should have been included with the blue group. For 33 it is divisible by 3 & 11 but $3a-3b$ or $a-b$ would work but this is another rule for 11!. For the numbers 47, 67 and 97 the rules are $3a+5b$, $3a+7b$ and $3a+10b$. for 27 we have $3a+3b$ and also $a-8b$.

Then I tried to explain how these rules come about with some success. For all of these we try to multiply $10a+b$ by a number so when we add or subtract the rule we get a number that is divisible by the original number.

The work is shown below. For example for testing for the divisibility by 31 we do the following:

$3(10a+b)=30a+3b$ so the rule becomes $a-3b$ then if we add **$30a+3b$** to **$a-3b$** we get **$31a$** that is always divisible by 31. All the rest follow a similar pattern that we went over. See below:



$$2 \times 7 \times 71 = 994$$

They found 2 and 7 ($99 - 8 = 91 = 13 \times 7$) quickly then they knew that the two digit number should end in 1 to get a 4 for the last digit of 994. They realized 81 was too large so it had to be 71. I would make them test the rule for 71 for example here $a - 7b$ would give them $99 - 28 = 71$. Then they would roll the dice and if they got 5 they would get $71 + 2 + 7 = 80 \times 5 = \400 .

Again we had a great session and we ended it a little after 3.



Math Circle 14 12/14/2022 1:50 to 3:00 pm

Cyrus, Marc and Dar

We worked with the $1 \leq 10$ machine for this Circle. I explained very briefly how it worked and gave them a multiplication problem to do on the machine and not on paper or in their head. Marc was challenged more than Cyrus for using the machine as he is fluent with the different algorithms for multiplication and division but Cyrus was never introduced to these algorithms so he was more willing to try the $1 \leq 10$ machine first.

My goal was to show them exactly how the algorithms worked so we had a bit of challenge and also some competition between them. Never the less we proceeded and watched the video [The Power of Place Value](#) and did the problems he was doing by stopping the video and restarting it. The video runs for 30 minutes but with the stops and Cyrus and Marc doing them the problems it took longer. When we got to showing numbers in base 2 or base 3 with $1 \leq 2$ and $1 \leq 3$ machine and even the $2 \leq 3$ Machine Marc became more interested. We also watched the sections on addition and subtraction and multiplication.

Finally I also started the video [Pushing Long Division to New Heights](#) but we could only watch the very first part and then I jumped to the section showing how the long division was done on the machine and how it clarified what was happening in the long division algorithm by comparing it to what they saw on the machine. Again Marc wanted to do the problem with the standard algorithm and I wanted them to be impressed by seeing how the $1 \leq 10$ machine actually shows how the algorithm really works. For Cyrus it was an introduction to the Long division algorithm and it is curious that when the next day he did a couple of problems both on the machine and using the algorithm he just wanted to do it with the algorithm just like Marc! I think that even for Marc it was good to see the relationship.

All in all this was not nearly as exciting for the boys who have been spoiled with actually playing games as this was a more formal session. Never the less I think it benefited both of them a lot for visualizing numbers in base 10 and other basis (we had already used the machine for base 2 in a previous session) in addition to see how the standard algorithms work and how working from left to right instead of right to left maybe even easier for all algorithms. I hope what they walked away from is that all algorithms are

OK but you need to also be able to see why they work and not just memorize the algorithm.

The following is the division problem 3,888 divided by 12. The answer is found by counting the number of 12 patterns at different levels.



We started with the $1 \leftarrow 10$ machine again but switched to the $1 \leftarrow X$ machine quickly. We covered a lot of very advanced topics as you will see but it went quite well. Marc who was not very keen on using the $1 \leftarrow 10$ machine last time was much more interested when he saw what the $1 \leftarrow X$ Machine does and how it is really the generalized form of $1 \leftarrow 10$ Machine or any other Machine like the $1 \leftarrow 2$ or $1 \leftarrow 3$ Machines. I followed mostly what is in *Lecture 7. Pushing the Long Division to New Heights* and *Lecture 8. Pushing Long Division to Infinity* from [*The Power of Place Value*](#). They were both quite impressed with what they saw and did.

I started them with 3 challenge calculations hoping that we would get to them by the end of session but we did not quite make it;=<

The problems were $1/(1-1/2) - 1$ and $1/(1-1/3) - 1$ and $1/(1-1/4) - 1$ I gave them 5 minutes alone and they were struggling with them even though they could get the some of the answers. Writing and simplifying the fractions was not easy for them. Cyrus is a bit more organized when writing problems and the steps to get to the solution but in this case they both had a hard time trying to write the intermediate steps. I did show them that for example $\frac{3}{4} / \frac{1}{2}$ becomes $6/4$ and they both eventually found that the solutions are 1 , $\frac{1}{2}$ and $1/3$.

Next I gave them the same problem they had seen 276 divided by 12 with Marc doing it on the white board and Cyrus on the $1 \leftarrow 10$ Machine. Marc got the answer that was 23 faster and was not that interested in seeing the solution on the Machine.

Then I asked them to solve the following problem $2X^2+7X+6 / X+2$. Marc was stuck but Cyrus did it quickly on the $1 \leftarrow X$ Machine (I had shown him how this worked) and showed that the answer was $2X+3$. This was a set up to impress Marc and it worked;=> In reality the two problem are exactly the same see the images below: Then I had them verify that if $X=10$ (Base 10) they are solving $276/12 = 23$ and if they are working with $X=2$ they are solving $(8+14+6)/(2+2)=28/4=7$ and if $X=3$ then they are solving $45/5=9$ (Marc was now really into it and got this very quickly!

$$(2x^2 + 7x + 6) \div (x + 2) = x + 3$$

$$(2x^3 + 5x^2 + 8x + 3) \div (2x + 1) = x^2 + 2x + 3$$



Next I had them solve $2X^3 + 5X^2 + 8X + 3 / 2X + 1$ on the 1 $\leftarrow$ X Machine and now they were both working on it and firing on all cylinders;=> and found the answer that was $X^2 + 2X + 3$ and came up on their own that if $X=10$ then this was $2,583/21$ and the answer was 123.

Next I said let's do a harder problem and gave them $X^2 - 1 / X - 1$ to solve on the 1 $\leftarrow$ X Machine. The issue is that negative numbers are shown with an orange cube and now the combination of a regular cube and an orange cube is not in the dividend. For more images refer to the Lecture 8 video and the image below. They were able to put a regular cube and orange cube in the X area (as these will cancel each other) with some help from me and find the answer as $X + 1$ and I showed that this will mean $X^2 - 1 = (X - 1)(X + 1)$ (I am not sure if they got this completely).

However, they saw that $X^3 - 1 / X - 1$ will be $1 + X + X^2$ and $X^6 - 1 / X - 1$ will be $1 + X + X^2 + X^3 + X^4 + X^5$ and in general $X^6 - 1 / X - 1$ will be $1 + X + X^2 + X^3 + \dots$

$$\frac{x^2 - 1}{x - 1} = x + 1$$

$$(x^2 - 1) = (x - 1)(x + 1)$$

$$\frac{x^3 - 1}{x - 1} = x^2 + x + 1$$

$$\frac{x^6 - 1}{x - 1} = x^5 + x^4 + x^3 + x^2 + x + 1$$



$$\begin{aligned}
 X^6 - 1 &= (X-1)(X^5 + X^4 + \dots + X + 1) \\
 \frac{X}{2} &= 4 \quad X = 2 \times 4 = \\
 \frac{X}{5} &= 6 \quad X = 5 \times 6 = \\
 100 \\
 X^{100} - 1 &= (X-1)(X^{99} + \dots + X + 1) \\
 &\quad \text{Something}
 \end{aligned}$$

Then I told them that now they could solve the problem that I had given them before "Is $32^{100}-1$ a Prime Number?"

They could see that $32^{100}-1=(32-1)(\text{Something})$ so it had a factor of 31 and not Prime the same way Any number to any power minus 1 will not be a prime. They made fun of "Something" but I told them that this is a long polynomial and it is not important for this problem what it is. I also gave them a few other examples.

$$\begin{aligned}
 X &= 2 \\
 100 \\
 32 - 1 &= 31 \times \text{something} \\
 X^{100} - 1 &= (X-1)(\dots) \\
 52 &\quad (4^5 + 4^4 + \dots + 1) \\
 9 - 1 &= 8 (\text{something}) \\
 5 \\
 2 - 1 &= 1 \times 31 \text{ Prime} \\
 27 \\
 2 - 1 &= 1 \times 27 \text{ Prime}
 \end{aligned}$$

Then I asked them about 2 to different powers minus one. Now they saw that these may be prime or not and I told them about the Mersenne Numbers and how in 400 years they have found only 49 of them and if they found one they can become a famous mathematician!

$$\begin{array}{ccccccc}
 2 & 4 & 8 & 16 & 32 & 64 & 128 \\
 1 & (3) & (7) & 15 & (31) & 63 & (127) \\
 \\
 & \text{Mersenne 49} \\
 2^{11} - 1 & = 2047 = 23 \times 89 \\
 2^8 - 1 & = (2 \times 2 \times 2 \times 2)(2 \times 2 \times 2 \times 2) - 1 \\
 & = 16^2 - 1 \quad x=16 \\
 & = (x-1)(x+1) \\
 & = 15 \times 17
 \end{array}$$

They wanted to go and play and they deserved it. It is amazing that they put up with this much sophisticated material for 55 minutes. I attribute it to the magic of the 1 $\leftarrow$ X Machine. I have seldom been fascinated with any visual tool than this Machine. I also asked Marc to share the polynomial divisions with Hugo and I showed him the machine. I strongly recommend that Hugo watches The Power of Mathematical Visualization when he finds the time it will give him a new and powerful angle to look at (literally) many important Math problems. Hugo did do the long hand division of the polynomial as he probably reverse engineered it. Great way to go about it.

Math Circle 16 25/01/2023 2:00 to 3:30 pm

Cyrus, Marc and Dar

I wanted to review the work with the 1 $\leftarrow$ X machine in particular the derivation of the formula for the geometric series but since I had promised the boys to have them play and program Edison I put that on the back burner.

They worked with the 2 Edisons and programmed them using the Bar Codes and also using the TV remotes that I gotten for them. So they could make their Edisons move back and forth and turn and spin right and left and Beep and make music. Then they started to play a soccer game with their Edisons. It was fun but a little too hard to control the ball and get behind the ball at the beginning but they enjoyed it never the less. They also started pushing each other around (with their Edisons) and they enjoyed that even more.

Next we came back and talked about programing Edison. This was more familiar for Marc as he has a similar robot at home but it was a good intro for Cyrus also who got busy playing with the Lego pieces and his Edison except when I should him that he could play real music with it and he wanted to program a long set of notes but finally settled for a shorter version. .

They both saw how we programmed Edison and how it could execute the program afterwards. Marc experimented more with the demo programs. I have been trying to enable at least one of the the iPads also to be able to program Edison but have not been successful so far for a couple of different reasons. For now they can program one at a ttime on my PC. I will keep trying so they can program side by side in a future session.

All in all a good long session and a good start for introducing them to programing. We will continue this in another session.



Math Circle 17 25/01/2023 2:00 to 2:55 pm

Cyrus, Marc and Dar

We worked on finding the cube roots of cubes generated from 2 digit numbers. We also started looking at the square roots of squares generated by 2 digit numbers. I had worked on this with Cyrus some time back and he had gotten pretty good the cube root but square roots were a bit harder for him and Dar.

First I had both of them complete the table that allows you to find the cube roots see below. Then I discussed how this one table can let them find the 10s-digit and the 1s-digit for the cube root of the number that is given to them. I asked Cyrus also to explain the table as best he remembered to Marc and he did a good job.

Then we looked at some examples for Marc and he was very quick to see how everything worked. Then we started to do head to head competition with who ever found the answer first just raising their hand. First Cyrus and Marc then Marc and Dar and Finally Cyrus and Dar. Cyrus was the fastest and Mark who was looking at the cheat sheet stated to get faster and faster.

Cyrus took the tables and we continued. They were both very excited and were jumping up and down in particular when they beat Dar. Cyrus who beat everybody almost every time but Marc got faster and faster and even Dar could be faster than Cyrus once in while. Marc started to hold his own against Dar at the end.

Then I showed them the Table for the square roots and how this was a bit harder because they had to make a choice for the 1-s digit depending on the size of the number. For the cube root there is always a unique answer depending on the last digit of the number.

We stopped here and will pick up the square roots next week. They really enjoyed this and Marc took his table and was charged to show off at home for Hugo and his parent;=>

That night I made another table for the square roots that I am attaching here and will share next week with them.

Lookup Tables for Square Roots

X	X ² (00)		X	X ²	These numbers will be used to decide between: 1&9 - 2&8 - 3&7 - 4&6
1	1				
2	4		15	225	
3	9		25	625	
4	16		35	1225	
5	25		45	2025	
6	36		55	3025	
7	49		65	4225	
8	64		75	5625	
9	81		85	7225	
10	100		95	9025	

For example for **2116** the tens digit is 4 and
1s digit is 6 since **21** is greater than **20**

Lookup Table for Third Power of 2-Digit Numbers

X	$X^3(000)$
1	1
2	8
3	27
4	64
5	125
6	216
7	343
8	512
9	729

Math Circle 18 02/08/2023 2:00 to 3:00 pm

Cyrus, Marc and Dar

We started with some Mathmagic from Secrets of Mental Math by Arthur Benjamin Pages 200 to 206. Then we worked on taking square roots of double digit numbers.

First we did the magic of 1089. They liked the trick and they started speculating about why it worked even before we had finished it. Cyrus said it worked because the intermediate numbers were divisible by 3. Then they ended up with the intermediate numbers being divisible by 9. And also 1089 being divisible by 9 and Cyrus said it was 11 times 99 that is correct. Next I tried to explain why the trick worked. The first part was not that easy but they had no trouble with the algebra part of it. In the middle of it I saw that I had forgotten the punch line and had to look it up again! We wrote it on the board (see below) and they were impressed and we discussed it some more. I wanted them to try another number each but they wanted to move on. I noticed that Marc's autograph has improved a lot and he is much neater than before after I had told him that this was important;=> I had prepared the missing digit trick but decided that I may screw it up so I skipped it for now.

Next we did the Leapfrog Addition. (see left side of picture below) and their work side by side. They both immediately said this was a Fibonacci sequence. Of course it can be if you start with 2 consecutive Fibonacci numbers and the method is similar and it is cool that they see the pattern immediately. They both had a little trouble making the totals that took away from the magic of the trick but we checked each one and got the right totals and went over why the 7th term times 11 is equal to total. This covered a good bit of Algebra but they followed it pretty well. We also did the second part of the activity that ends with 1.61 for the first 3 digits when you divide the tenth term by the ninth term and they liked it but I skipped the proof that is too involved (see page 205). Again they wanted to move on instead of doing another example so we moved on.

Then I gave them the lookup table for square roots that Cyrus had seen before and did not want to use. I also explained it more for Marc on how we found the 10s digit (that is similar to the cube root but uses the 100s part of a number instead of 1000 part of the number for cube root) and also the 1-s digit that is more complicated as you need to decide between 1,9 or 2,8

or 3,7 or 4,6 based on the 100s part of the number (this makes finding the square roots more difficult than finding the cube roots)!

We started the races with asking Cyrus to raise his hand if he figured out the answer as he did it most of the time a lot faster than Marc who had just seen the method and me also (old brain). Marc and I used our cards at first. First Cyrus and Marc then Marc and Dar and Finally Cyrus and Dar was the order of competition and then we kept on going. Cyrus was the fastest and Mark who was looking at the cheat sheet stated to get faster and faster. I was getting better also. We did this for a good 25 minutes and they were very excited in particular when they beat me like last time and Marc got better and better and we could have gone on but it was after 3:00 o'clock and we decided to stop.

It was again a very good session and Marc wanted to take his card which I had made for him and when I proposed that he take a card for Hugo he said no;=> In the next session he said Hugo had figured out the process by himself very proudly (this is indeed not an easy thing to do).

Lookup Tables for Square Roots

X	X ² (00)		X	X ²	These numbers will be used to decide between: 1&9 - 2&8 - 3&7 - 4&6
1	1				
2	4		15	225	
3	9		25	625	
4	16		35	1225	
5	25		45	2025	
6	36		55	3025	
7	49		65	4225	
8	64		75	5625	
9	81		85	7225	
10	100		95	9025	

For example for **2116** the tens digit is 4 and 1s digit is 6 since **21** is greater than **20**

$$\begin{aligned}
 abc &= 100a + 10b + c \\
 -cba &= 100c + 10b + a \\
 \hline
 &= 99a - 99c \\
 &= 99(a-c)
 \end{aligned}$$

198, 297, 396

495, 594, 693, 792

891

$$\begin{array}{r}
 693 \\
 396 \\
 \hline
 1089
 \end{array}$$

$$\begin{array}{r}
 891 \\
 350 \\
 \hline
 693 \\
 693 \\
 \hline
 1089
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We worked with the Pythagorean Theorem and Pythagorean triplets.

I started by reviewing the area of a rectangles with sides of 3 and 4 and they quickly said 12 and also for the triangle with sides 3 and 4 and they said 6. Then we worked with a rectangle and triangle with sides a and b and they came up with ab and $ab/2$ the second after a bit of back and forth. Then I had them draw a right triangle with sides 3 and 4 and measure the hypotenuse. They both got 5 (Cyrus said it was not exactly 5) but he agreed that it may be just the drawing. Then I had them draw a right triangle with sides 6 and 8 and measure the hypotenuse and they said it was 10 even before they were done measuring. Next I asked them if they could figure out how we could find the hypotenuse from the two legs if they were 3,4 or 6,8 or 9,12 and after some discussion I suggested that the formula maybe $c^2=a^2+b^2$ they checked the numbers and were impressed then we talked about all the triples that we could get from 3, 4, 5 and they went on and on.

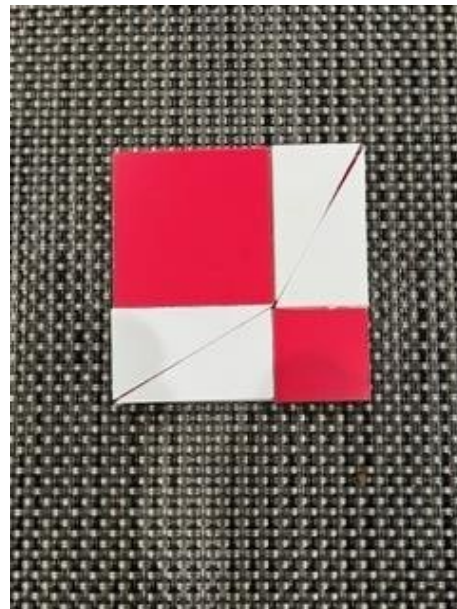
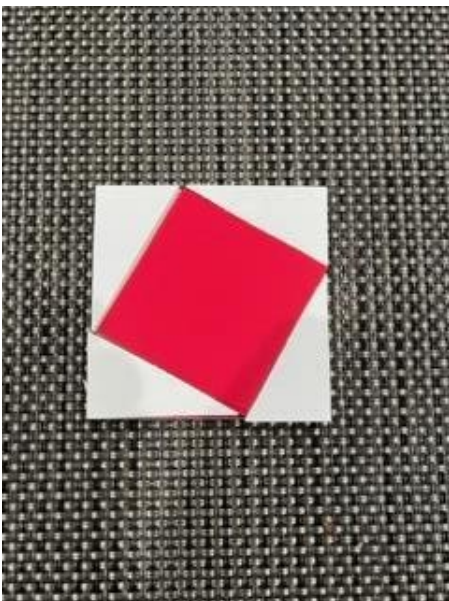
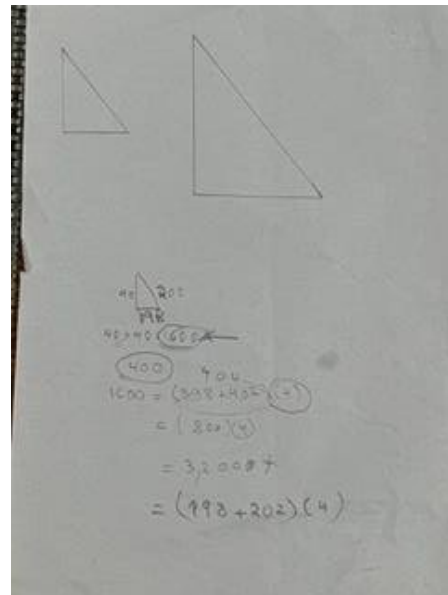
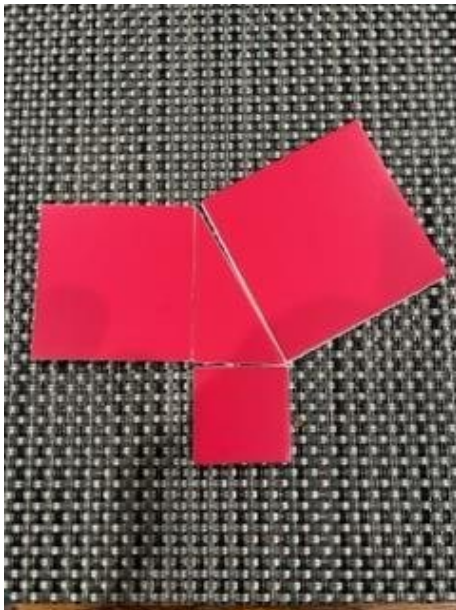
Next I gave them my puzzle (this is very old puzzle for the proof of the Pythagorean Theorem). As you can see in the pictures it shows that we can cover a square with sides $a+b$ either with a^2+b^2+2ab or c^2+2ab removing $2ab$ from both of these we can get $c^2=a^2+b^2$. In addition we see that this is a visual proof for the identity $(a+b)^2 = a^2+b^2+2ab$.

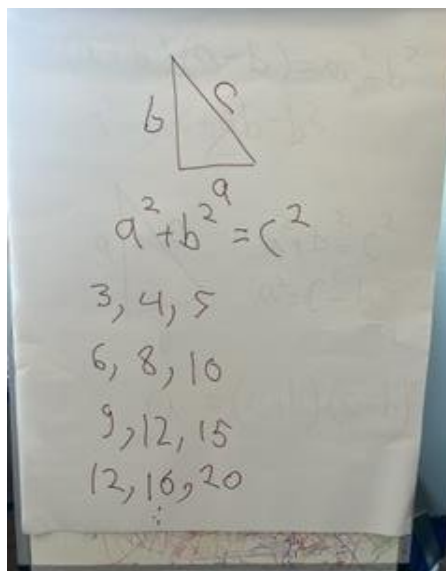
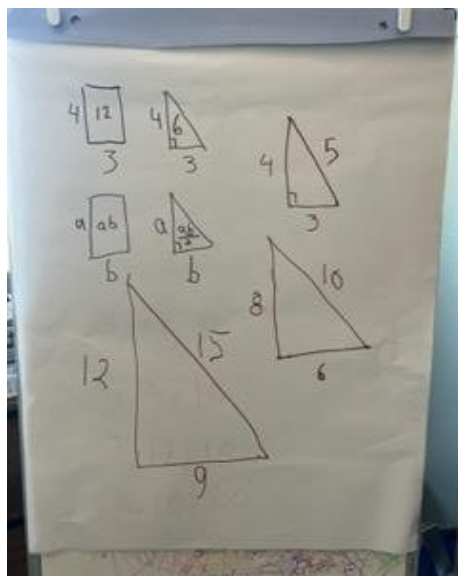
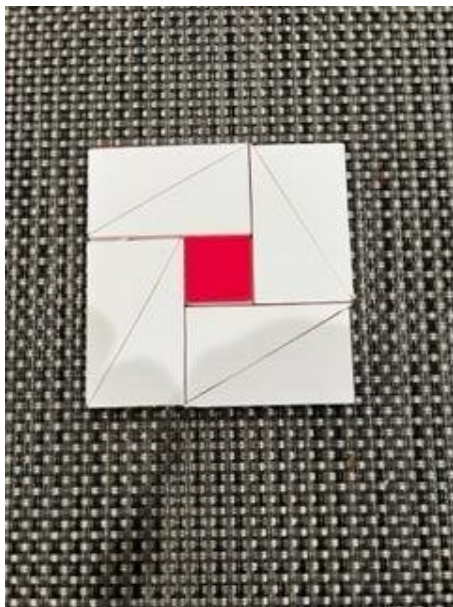
They played with the pieces and could see both of these proofs. At the end of the session we also started to look at the proof for $(a-b)^2 = a^2+b^2-2ab$ using the same manipulatives and a square with sides $(a-b)$ but we ran out of time.

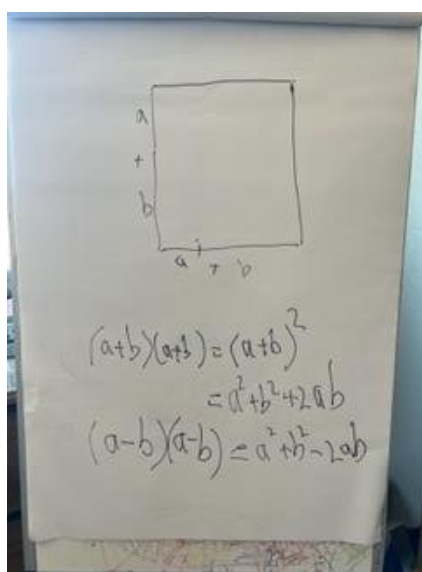
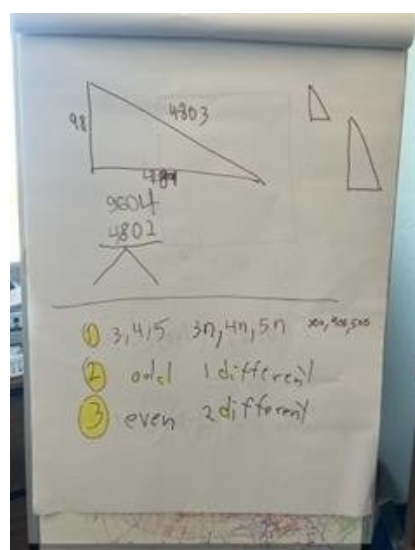
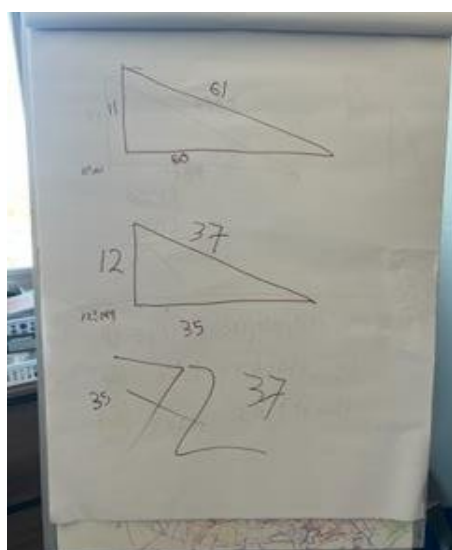
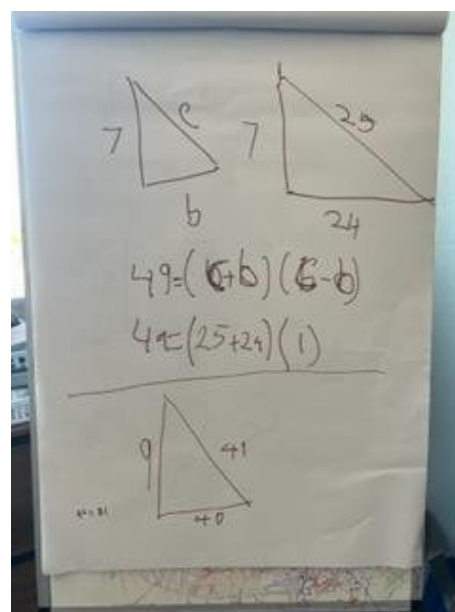
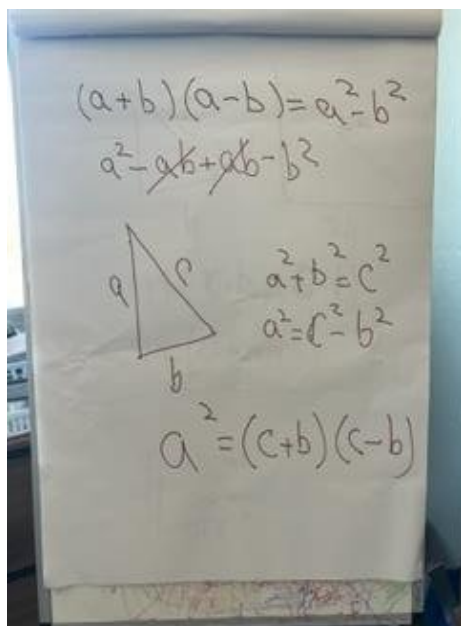
Next using the identity $c^2=a^2+b^2$ I showed them that then $a^2 = c^2-b^2$ and also that $c^2-b^2=(c+b)(c-b)= c^2-cb+ cb-b^2$ and using the identity $a^2= (c+b)(c-b)$ we can always find a^2 is odd find $c=b+1$ and if it even $c=b+2$. With did a few examples and they would come up with the solutions and would check them on the calculator. They were impressed. I told them maybe we can come up with a more general result that will be our discovery. I have included the out line for a general proof below at the end.

This was also a good session but I think we need to do another session for them to completely grasp the idea of how they can find Pythagorean

triples. I would like to also tell them more about the history of this theorem and how a version of this was known in the old Babylonian Period some 1,700 years before Pythagoras!

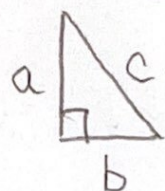






Possible Discovery of Var, Cyrus & Marc

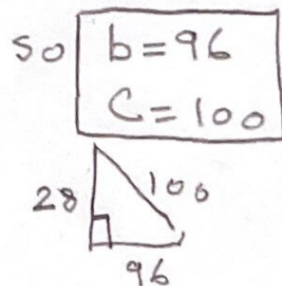
If one of the legs a of a right triangle is even (or odd) number and is divisible by an even (or odd) number d then we can find b such that $c = b + d$ where b is the other leg and c is the hypotenuse



$$\begin{aligned}a^2 + b^2 &= c^2 \\a^2 &= c^2 - b^2 \\a^2 &= (c+b)(c-b) \\&= (2b+d)(d)\end{aligned}$$

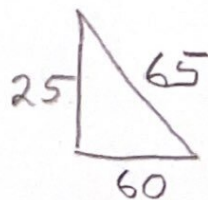
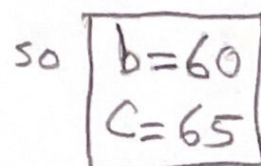
For example

$$\begin{aligned}\text{if } a=28 \text{ then } a^2 &= 784 \quad d=4 \\d=4 & \\&= (196)4 \\&= (100+96)4\end{aligned}$$



$$\begin{aligned}\text{if } a=25 \text{ then } a^2 &= 625 \\d=5 &\end{aligned}$$

$$\begin{aligned}&= (125)5 \\&= (65+60)5\end{aligned}$$



Math Circle 20 03/01/2023 2:00 to 3:35 pm

Cyrus, Marc and Dar

We continued the work with the visual proof (or demonstration really) of the Pythagorean Theorem and three other essential algebraic identities:

In a right triangle $c^2 = a^2 + b^2$

In General

$$(a+b)^2 = a^2 + b^2 + 2ab$$

$$(a-b)^2 = a^2 + b^2 - 2ab$$

$$a^2 - b^2 = (a+b)(a-b)$$

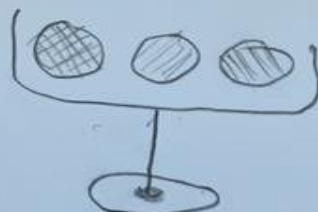
I thought that what we covered last time was very advanced for them so they needed more hands on activity to see the proofs.

I started them with a challenge below and poker chips of 3 colors to solve the 3 scoop ice cream problem below. I left them for 5 minutes first to discuss and try to come up with a solution.

An ice-cream shop has 3 flavors of ice-cream: Chocolate, Vanilla, Strawberry. It make triple-scoop ice-creams. The scoop can be of one flavor, two flavors or all different.

Please help them find how many different triple-scoop ice-creams they can make. The order does not matter.

Please also find how many scoops of each flavor do they need to make all the different triple-scoop ice-creams.



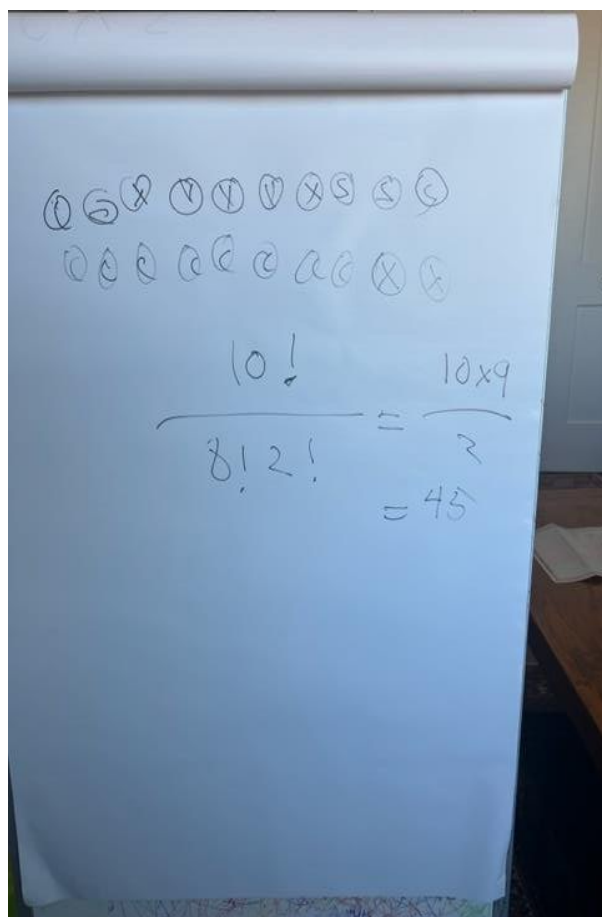
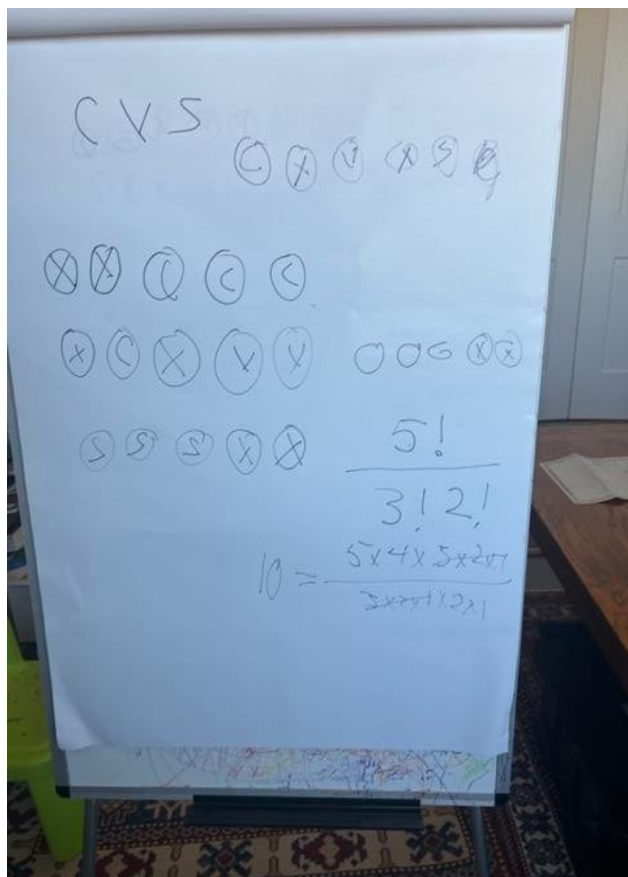
Initially they came up with 6 solutions just reordering a single flavor. After discussing that all these were really the same solution as order did not matter. Then they started throwing answers. I helped them make all the possibilities and agree that there were 10 solutions and also they needed ten of each flavor to make all the solutions.

[The both need to learn to slow down and examine their thinking and logic and also be able to discuss solutions with each other. I feel they have made a lot of improvement in the Math Circle on this but there is a lot of room for progress. (Let's not forget they are only 8!)]



Next I asked them if they could think of a way (hard question that we will cover in the next session) to see why the answer was 10 and what the answer would be if the we looked at 4 or 5 or 10 scoops of 3 flavors.

I should them how if we made 5 circles and used C, V, S for the flavors and put Xs in the other two circles they could come up with all the possibilities. They helped me come up with the correct solution for 3 of the same flavor. Then I told them that the solution was $5!/2! = 10$ (Cyrus had seen this before I am not sure Marc had seen it, but I explained it and we will cover it again next time). They had no trouble coming up with $10!/8! = 45$ for 10 scoops problem.



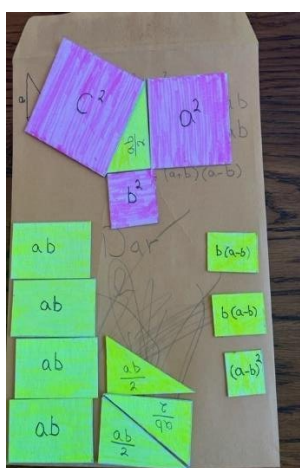
Next I showed them the equalities they were going to prove with out the right had side filled in and distributed the card boards that I had cut for all 3 of us and asked them to use 2 colors for these and a sperate color for the cadre.

The coloring took a little time but I think it was good as it slowed them down and while coloring we talked about the area for each of the pieces that we had covered in the last session. Initially none of the areas were written on our papers and this was added as we looked at each problem. I think they enjoyed the fact that I was also doing the coloring and I kept them entertained as they were coloring and asked some question. Rooz also came in and did some of the coloring for me and enjoyed it. Ava also got in at the end and did some drawing on my envelope;=>

The session went really well and after an initial complaining about coloring taking too long they got into the activity. Even Rooz did the first proof for the Pythagorean theorem (without knowing that he was doing the proof!) They were both seeing the proofs pretty well except for the proof for

$(a-b)^2 = a^2 + b^2 - 2ab$ which is not easy to see. I came up with another proof later and showed it to Cyrus that I show here which I think is a much better visual proof.

The session took more than an hour and a half but by the end they were completely engaged. A really good session. They still need some review of basic algebra that I will do for them. But I think they really enjoy doing work that is normally done in high school.



$$a^2 + b^2 = c^2$$


$$ab = (a+b)^2 = a^2 + b^2 + 2ab$$

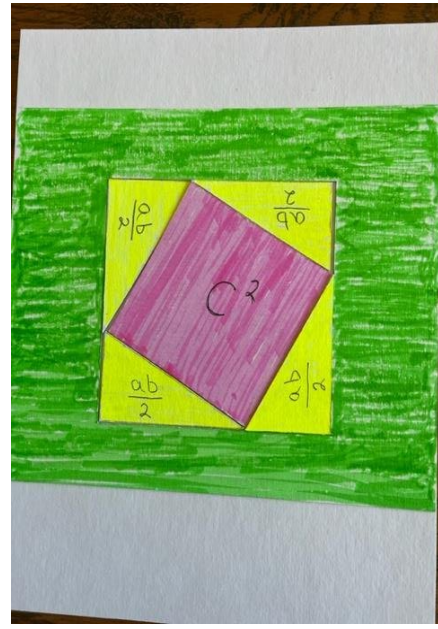
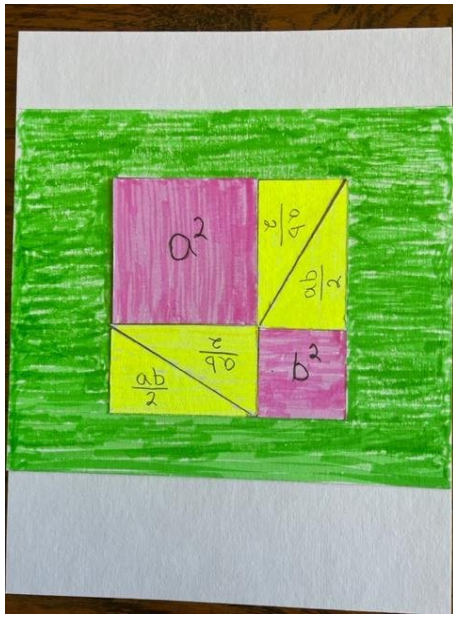
$$-(a-b)^2 = a^2 + b^2 - 2ab$$

$$a^2 - b^2 = (a+b)(a-b)$$

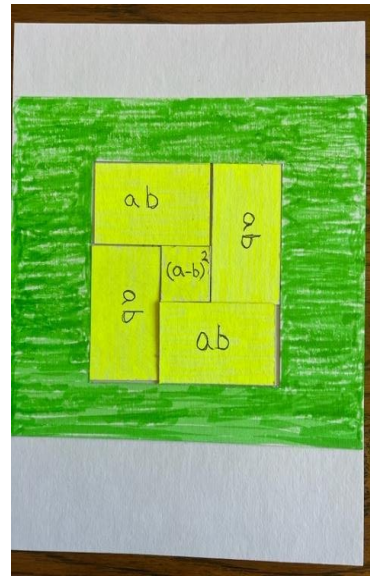
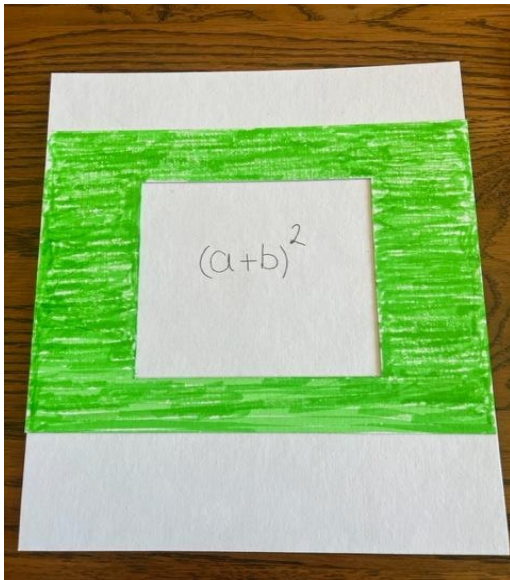
$$(a-b)^2 + 4ab$$

$$a^2 + b^2 + 2ab$$

We started to look at the proofs and we all started putting in the area for each of the Squares, Rectangles and Triangles including the big square $(a+b)^2$.

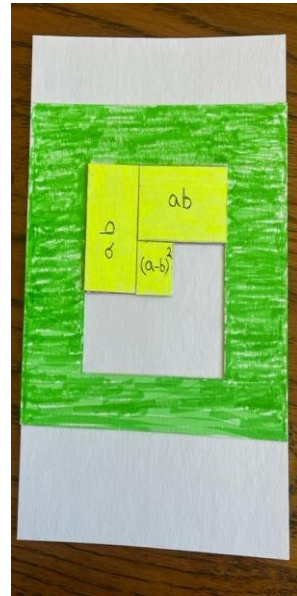
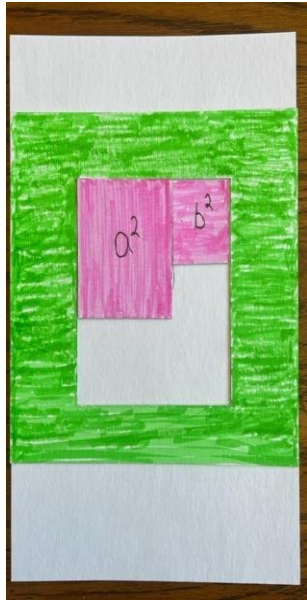


Visual proof of $c^2 = a^2 + b^2$ as they both fill $(a+b)^2$ and each has $2ab$ extra. Also we have a visual proof for $(a+b)^2 = a^2 + b^2 + 2ab$ from the first image above. They both did it and explained it.

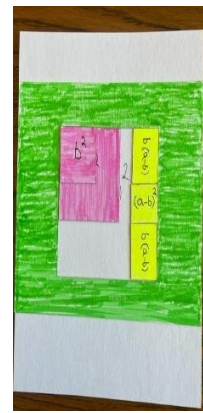
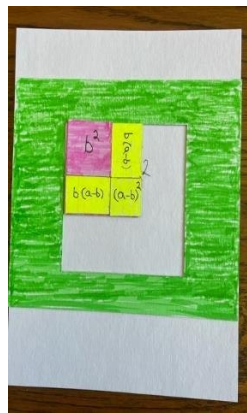
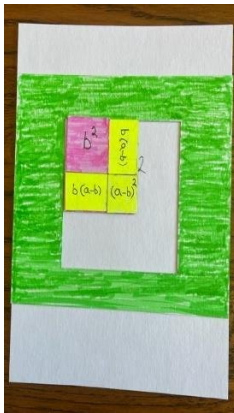


Comparing the square $(a+b)^2 = a^2 + b^2 + 2ab$ and the square $(a-b)^2$ one can see that if we take away $4ab$ from $(a+b)^2$ we get $(a-b)^2$ so

$$(a-b)^2 = a^2 + b^2 - 2ab$$



A better visual proof for $(a-b)^2 = a^2 + b^2 - 2ab$. If you cover (take away) $2ab$ from $a^2 + b^2$ you are left with the square $(a-b)^2$.



A visual Proof for the algebraic equality $a^2 - b^2 = (a+b)(a-b)$. by placing $-b^2$ on top of a^2 - we see what is left is $(a+b)(a-b)$.

Math Circle 21 03/08/2023 2:00 to 3:00 pm

Cyrus, Marc and Dar

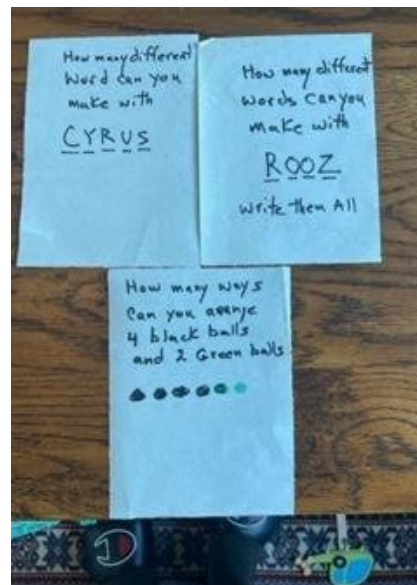
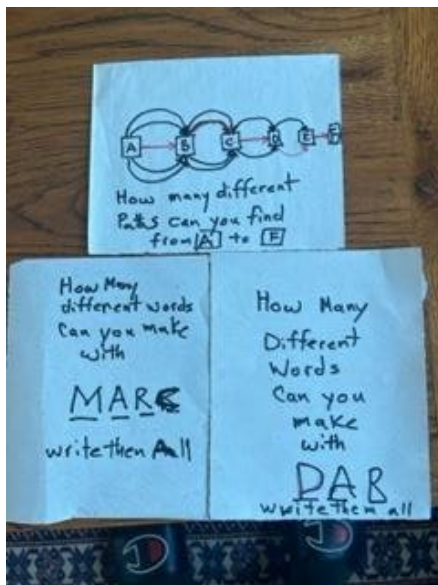
In his session we followed the lecture on Combinatorics the [Art of Counting from the Power of Mathematical Visualization](#). I came up with my own examples that are below. We did them as challenges and discussed them after they worked on them. The goal was to get them to the general formula that can solve even the problem of how many ways you can arrange the letters in BOOKKEEPER. The answer being $10!/3!2!2!1!1!1!$ and also the problem of Mrs. Baser choosing a math team that has Cyrus and Marc in it and 4 other students in their class (a really hard problem). Assuming there were 20 kids in their class the answer putting Cyrus and Marc aside is

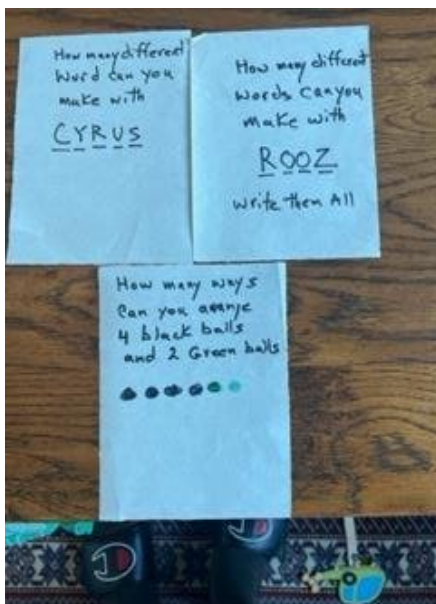
$$18!/4! = 266,765,571,072,000 \text{ over 266 Trillion teams;}=<>$$

This problem I did for them in Excel and I showed them how easily Excel can do this. Marc has some familiarity with Excel but Cyrus had not worked with Excel they were both impressed.

I have shown some of their work below (I only have Cyrus's work as Marc took his sheet) They were both engaged and this is a great introduction to solving all Combinatorics problems and I recommend that they watch the video also. I may play it for them later on.

All in all a good session and now they are ready to take on some of the interesting patterns for Pascal's Triangle in the next session.





Math Circle 22 03/22/2023 2:00 to 3:00 pm

Cyrus, Marc and Dar

In this session we worked on Pascal's Triangle (Omar Khayam's Triangle) . I used the Lecture on [Visualizing Pascal's Triangle from The Power of Mathematical Visualization Series](#) and Also the book [Visual Patterns in Pascaal's Triangle by Dale Seymour](#).

Initially I asked them to find the number of paths from A to B on the following sheet. We discussed it some and how it was related to what was covered in the previous session with combinatorics.

Essentially that they had to find the number of words with 4 Ds (Downs) and 4Rs (Rights) the one that is shown is RRRRDDDD. So the number is $8!/4!4!=70$. ThThen they filled the other cells first by doing the first row and first column by 1 and then the other cells. Cyrus he got that quickly and they are learning that they can simplify and then divide rather than multiplying all the numbers in the numerator and the denominator and the dividing these big numbers. Cyus had seen the Pascal's Triangle before and said that this is the Pascal's Triangle. I showed them that if they turn the table 45 degrees they will see the Pascal's Triangle.

Then I gave them the Pascal's Triangle to fill up with 16 rows and this was a challenge as the numbers get big at theend. I think never the less it was a good exercise for them because they had to slow down and do the additions one by one. And the 3 of us filled the rest of the Triangle. As always Marc went really fast and Cyrus wrote slowly and but very carefully and beautifully he also insisted that he wanted to do all the additions himself. Marc used some of my numbers and then colored the Triangle for all those that are divisible by 2 (even numbers) shown below and also figured out that the sum of the rows gives us the powers of 2. Cyrus was a bit crossed because he was not done yet and did not want us to proceed before he was done. I slowed down a bit. Rooz came and colored mine and Cyrus started the coloring and took it home to complete (he is so meticulous in his writing and coloring that it takes for ever). If we could make a MarcCyrus everything will go better and I think they are influencing each other a little and that is great. We did talk about other sequences in there and Cyrus recognized the triangular numbers (That they call step squad).

Al in all not a bad session but I had hoped to cover more material but doing all theses rows and coloring took a long time. We will look at Powers of 2, Triangular, Square, Hexagonal and Pentaminal numbers in the Pascals Triangle in the next session

How many different paths are there between A and B?
You Can only move Right (R) or Down (D).

Remember if you have a word
with n Rs and m Ds then the
total Number of different words
you can make is $(n+m)!/n!m!$

	0	1	2	3	4
0	A	→	→	→	→
1					↓
2					↓
3					↓
4					B

→ R ↓ D

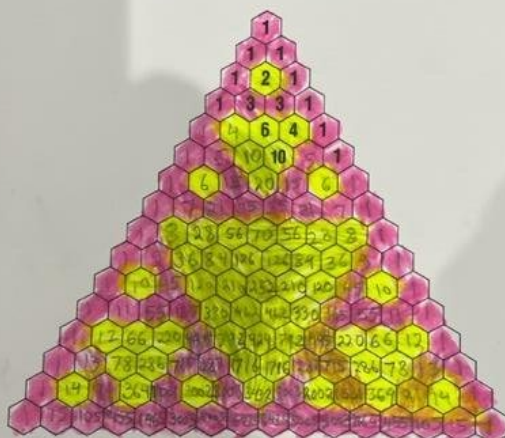
	0	1	2	3	4
0	1	1	1	1	1
1	1	2	3	4	5
2	1	3	6	10	15
3	1	4	10	20	35
4	1	5	15	35	70

Write the Number of
Paths in each Cell

What do you notice about the number of paths?

PASCAL'S TRIANGLE

Use the pattern to fill in missing numbers in Pascal's Triangle.



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$$\begin{array}{r} 70 \\ + 56 \\ \hline 126 \end{array}$$

$$\begin{array}{r} 155 \\ + 165 \\ \hline 320 \end{array}$$

$$\begin{array}{r} 286 \\ + 78 \\ \hline \end{array}$$

$$\begin{array}{r} 364 \\ + 91 \\ \hline 455 \end{array}$$

$$\begin{array}{r} 2 \\ 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 \\ \hline 4 \times 3 \times 2 \times 1 \times 1 \quad 4 \times 3 \times 2 \times 1 \\ 8 \end{array}$$

Math Circle 23-24 04/12/2023 04/26/2023 2:00 to 3:00 pm

Cyrus, Marc and Dar

In these two sessions we worked on Magic Squares and started looking at doing addition and subtraction from the left. I used mostly the material from Secrets of Mental Math pages 208-209.

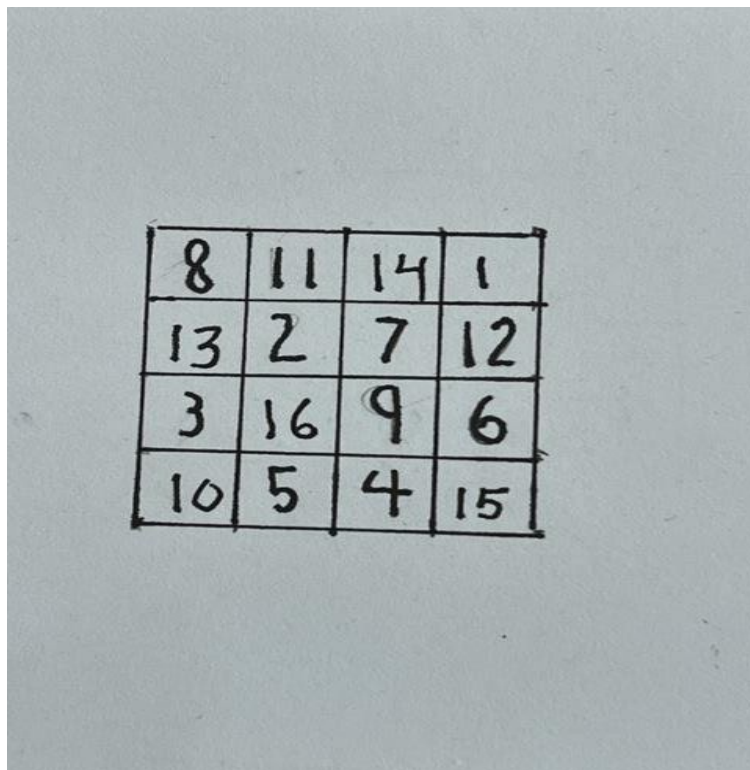
I initially gave them the following challenge:

On a 4x4 square grid and with 4 cubes of 4 different colors (16 cubes in all) arrange them so that there would only be one cube of a given color in any one row, column and diagonal. They liked the challenge and put in a good 10 minutes, and both came up with different solutions.

Next I gave them 16 cubes that were numbered 1 to 16 and asked them to arrange them in 4 different groups 1-4, 5-8, 9-12, 13-16 and use the color arrangements techniques to come up with a magic square that adds up to 34 for each row, column and diagonal. I had prepared a 4x4 grid that had some of the numbers marked on them to guide them.

This was a tougher challenge, and it took a little bit longer but again they were able to do it and come up with the same solution as the initial values I gave them will give a unique solution (I think). Then they filled in the numbers with pen on the 4x4 grid.

Next, we discussed why the sum should be 34 for each row, column or diagonal. And I had them look at the formula for adding $1+2+\dots+16 = 16 \times 17 / 2$ and this would be 4×17 or 4×34 so each row, column or diagonal has to add to 34 for a magic square of course.



8	11	14	1
13	2	7	12
3	16	9	6
10	5	4	15

Next I showed them the trick that if they were given a larger number than 34 say 50 then they could divide the difference by 4 and add it to each cell, In this case the difference was 16 so they had to add 4 to each cell to have a magic square that added up to 50 see below.

We discussed that if the difference was not exactly divisible by 4 then the remainder should be added to 4 cells in one of the groups. In the case of 35, 1 was added to the group 5-8 see below. Then they also

checked that they could add all the difference in any 4 cells as long as they are in a unique row, column and diagonal. For example for 78 we get $78 - 34 = 44$ and we can add 11 to 4 cells as shown below.

12	15	18	5
17	6	11	16
7	20	13	10
14	9	8	19

9	11	14	1
13	2	7	13
3	17	9	6
10	5	5	15

19	11	14	1
13	2	7	23
3	27	9	6
10	5	15	15

I started talking about doing addition from the left but by then they were done and wanted to play so we call it quits!

All in all a different but good sessions where they were talking to each other about the challenges a lot.

Math Circle 24 04/26/2023 2:00 to 3:00 pm

Cyrus, Marc and Dar

In this session we continued our work with Magic Squares.



I had made and laminated an initial Magic Square for each of us from last time. I worked with them also and suggested that if we switched or permuted the position of the number groups would we still have a Magic Square. We named these groups A:1-4, B:5-8, C:9-12, D:13-16 then to start with

Marc switched A < - > C Cyrus did A < - > B and Dar did A < - > D

then we all checked and found that the All resulting Squares were Magic Squares!

We continued with also switching the other two groups

Marc switched B < - > D Cyrus did C < - > D and Dar did B < - > C

in each resulting square and again we found that we had Magic Squares. They are really Magical!

Next we tried to permutating every and all groups by 1 so for example 1-> 2, 2->3, 3->4 and 4->1 and we found that again we had magic squares. Next, we tried permutating by 2 positions and 3 positions and we always ended up with magic Squares. Cyrus actually did a different permutation by switching

1< - > 4 and 2 < - > 3 instead of a 3 move and still had a magic square.

Finally, I wanted them to do a Magic Square that added up to 98 but they were tired and wanted to play. Cyrus tried and so did I but we both made a mistakes and added 16 ($98-34 = 64/4=16$) to just one group and decided we would complete it next time.

All in all these two session were great for them to do some discovery and research for non-trivial problems and discuss it also. They enjoyed it but did want to get to play at the end.

Marc's Work

	8	3	14	9	16	3	6	9	7	10	13	4	6	9	16	3
	13	10	7	4	5	10	15	4	18	1	2	17	15	4	5	10
	11	16	1	6	11	8	1	17	2	15	14	5	1	14	11	8
Marc	2	5	12	15	2	13	12	7	9	8	3	14	12	7	2	13
	5	12	15	2	17	24	27	17	8	11	14	1				
	14	3	8	9	26	18	20	21	13	2	7	12				
	4	13	10	7	19	25	22	19	3	16	9	6				
	11	6	1	16	23	18	16	28	10	5	4	15				

Cyrus' Work

4	11	14	5	4	15	10	5				
13	6	3	12	9	6	3	16				
7	16	9	2	7	12	13	2				
10	1	8	15	14	1	8	11				
7	10	13	4	6	9	16	3	5	10	15	4
16	1	6	11	15	4	5	10	16	3	6	9
2	15	12	5	1	14	11	8	2	13	12	7
9	8	3	14	12	7	2	13	11	8	1	14
24	11	14	1	8	11	14	1				
13	2	23	12	13	2	7	12				
3	16	9	22	3	16	9	6				
10	21	4	15	10	5	4	15				

Dar's Work

8	11	2	13	8	11	14	1
1	14	7	12	13	2	7	12
15	4	9	6	3	16	9	6
10	5	16	3	10	5	4	15
12	17	2	13				
1	14	11	8				
15	4	5	10				
6	9	16	3				
7	10	13	4				
16	2	6	11				
2	15	12	5				
9	8	3	14				
6	9	16	3				
15	4	5	10				
1	14	11	8				
12	7	2	13				
5	12	15	2				
14	3	8	9				
4	13	10	7				
11	6	1	16				

Math Circle 25 05/03/2023 2:00 to 3:00 pm

Cyrus, Marc and Dar

In this session we continued our work on Pascal's Triangle (Omar Khayam's Triangle). I used a couple of examples from the book [Visual Patterns in Pascal's Triangle by Dale Seymour](#).

First, we worked with Figurate Number Sequences: Triangular, Square, Pentagonal and Hexagonal Sequences and extended them further. Initially they wanted to mentally do the extension, and this got harder for Pentagonal and Hexagonal sequences. Eventually I managed to convince them that if they looked at the difference between two consecutive elements they could easily extend the sequences. They did this after some resistance, but I think will remember the technique that is quite useful for any sequence extension.

Next we worked on relating the Square, Pentagonal and Hexagonal sequences to the Triangular numbers. Cyrus quickly remembered that consecutive Triangular numbers add up to a Square number and this was one of his discoveries almost 2 years ago! Marc was trying his best and following along but he had a headache because of the bump that he got at school. Then Cyrus also added that Pentagonal and Hexagonal numbers could be made by doubling or tripling an element of the Triangular numbers and adding it to the next Triangular number. It was impressive as I had to think a lot more when I did the problem and I actually only saw the pattern that Marc found later.

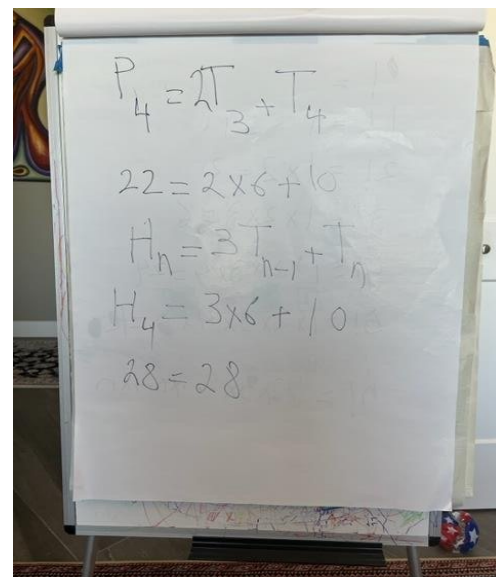
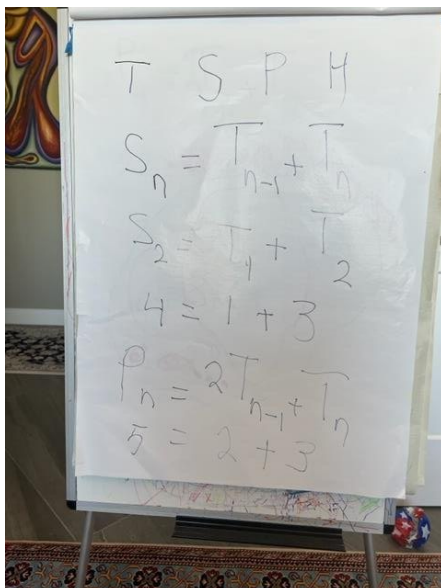
Next I showed them that we can use the symbols T for Triangular, S for Square, P for Pentagonal and H for Hexagonal numbers and in this way we can write what they had discovered as:

$$S_n = T_{n-1} + T_n$$

$$P_n = 2T_{n-1} + T_n$$

$$H_n = 3T_{n-1} + T_n$$

Below I show what we did. At first Cyrus made his famous comment: "It does not make any sense at all!" but when we went through the examples they both appreciated the power of the formulas and that it can be applied for any n.



Next they found and colored the Triangular Numbers on the Pascal's Triangle. Also they agreed that they could see the Square numbers as the sum of the consecutive Triangular numbers after some reflection. Marc also found that Hexagonal numbers are Triangular numbers when we skip one! The formula would have been:

$$H_n = T_{2n-1}$$

and also we could have concluded:

$$T_{2n-1} = 3T_{n-1} + T_n$$

But I missed the opportunity to point out their discovery. You may wish to discuss it with Marc or work on it your self. I may mention it next week quickly. I still can not prove the last equality! Very interesting find!

They also colored the Hexagonal numbers with coloring half of each cell. Then we tried to find the Pentagonal numbers in the Pascal's Triangle and it took a while and even Dar could not find them even though he had seen the pattern before. Finally they got the book and found the pattern but they both objected to the fact that the pattern started clearly with 12 and 1 and 5 were not obvious no matter how I tried to explain. Marc had a few suggestions before we found the pattern and took credit for predicting that it was a dog leg shape pattern.

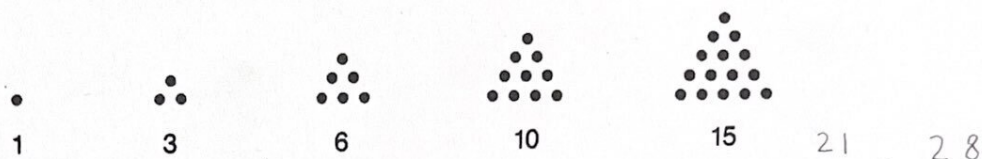
Next we did the King Strut's Cubes and Marc needed a talking to as he was fading but they both did a good job with drawing and finding the numbers and Cyrus shouted that the numbers came from the Pascal's Triangle. Of course Pascal's Triangle was right in front of them!

All in all it was a good last session as we will celebrate our year of Math Circle next Wednesday.

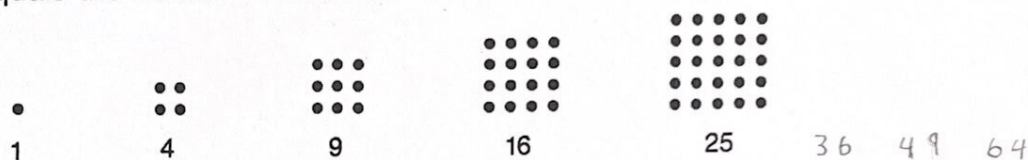
INTRODUCTION TO FIGURATE NUMBERS IN PASCAL'S TRIANGLE

The ancient Greeks were very interested in numbers that are connected by geometric forms. *Figurate numbers* (also known as polygonal numbers) are those numbers that can be arranged in a polygonal array.

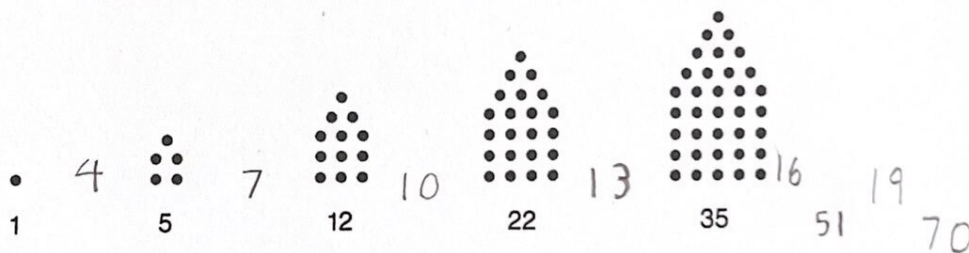
Triangular numbers can be represented by dots in a triangular array.



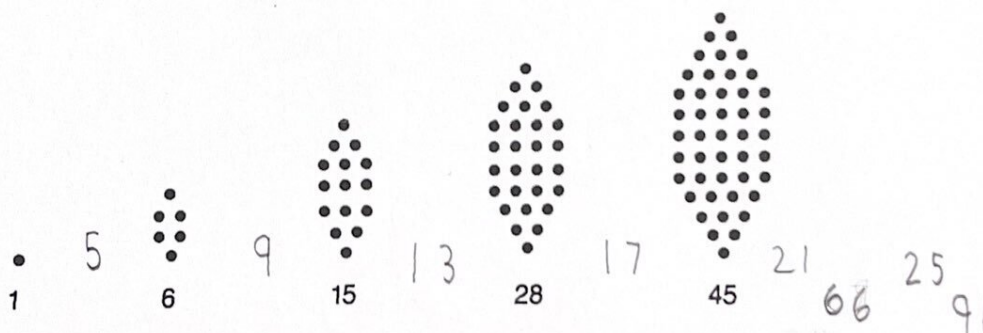
Square numbers can be arranged in a square array and are of the form n^2 where n equals the number of dots on a side.



Pentagonal numbers can be arranged in a pentagonal array.

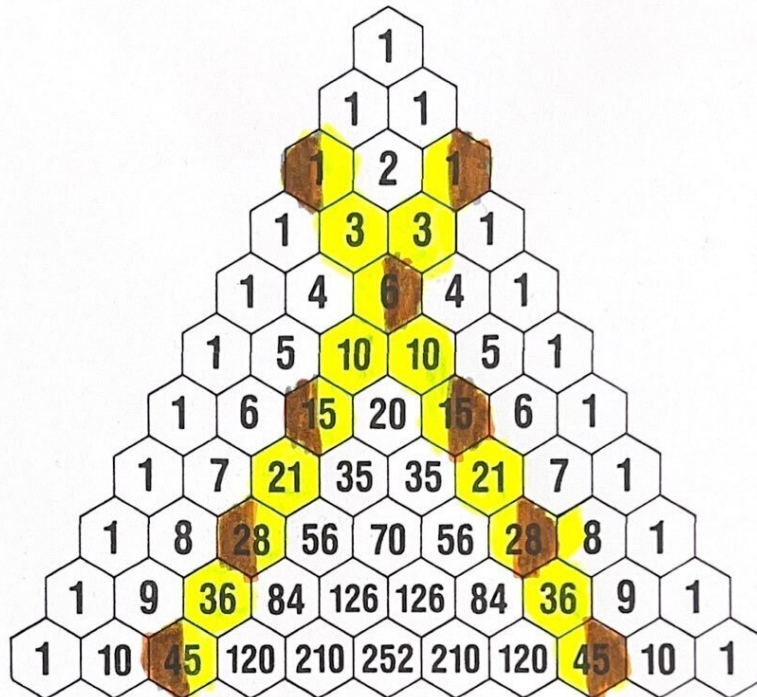


Hexagonal numbers can be shown as dots in an hexagonal array.



1. (a) What is the relationship between triangular numbers and square numbers?
- (b) Triangular and pentagonal numbers?
- (c) Triangular and hexagonal numbers?

POLYGONAL PATTERNS



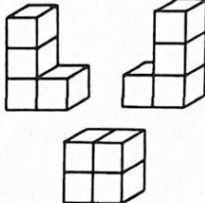
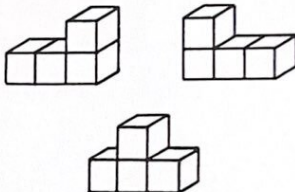
1. The square numbers 1, 4, 9, 16, 25, 36, ... can be found in at least two patterns in Pascal's Triangle. What are these patterns? (Hint: The patterns involve the sum of certain numbers in the triangle.)
2. Can you find a pattern for pentagonal numbers? If so, what is it?
3. Look for the patterns for triangular numbers and hexagonal numbers. What do the patterns have in common?

KING STRUT'S CUBES

King Strut kept his gold in cubes. He enjoyed handling his gold and often spent time stacking these cubes. The king only stacked cubes directly on top of each other or side by side (never one behind the other).

The diagram below shows the different ways of stacking four gold cubes.

The Different Ways of Stacking 4 Gold Cubes

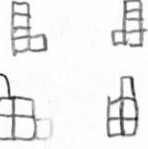
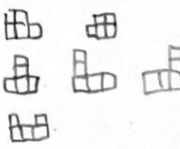
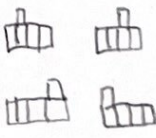
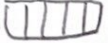
1 column	2 columns	3 columns	4 columns
			
1 way	3 ways	3 ways	1 way

Total = 8 ways

1. Make a diagram showing how three gold cubes could be stacked according to the king's rules.

		
1 way	2 way	1 way

2. Make a diagram showing how five gold cubes could be stacked.

				
1	4	6	4	1

3. Predict the number of ways six gold cubes could be stacked.

1	5	10	10	5	1
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4. What is the pattern?

